Linear Algebra

Lecture 14 - Quadratic Forms

Oskar Kędzierski

14 January 2018

Quadratic Form

Definition

A function $Q\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is called a **quadratic form** if

$$Q((x_1, \ldots, x_n)) = a_{11}x_1^2 + \ldots + a_{nn}x_n^2 + \sum_{1 \leqslant i < j \leqslant n} a_{ij}x_i x_j,$$

that is, it is a function given by a homogeneous polynomial of degree 2 in variables $x_1, \ldots, x_n$.

Examples

$$Q((x_1, x_2)) = x_1^2 - x_2^2$$

$$Q((x_1, x_2, x_3)) = x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$$

Symmetric Matrix

Recall

Definition

Matrix $A = [a_{ij}] \in M(n \times n; \mathbb{R})$ is called **symmetric** if $A^\intercal = A$, i.e. $a_{ij} = a_{ji}$ for $i, j = 1, \ldots, n$.

Example

$$\text{matrix } \begin{bmatrix} 0 & 2 & 5 \\ 2 & 4 & -3 \\ 5 & -3 & 1 \end{bmatrix} \text{ is symmetric}$$

$$\text{matrix } \begin{bmatrix} 0 & 2 & 6 \\ 2 & 4 & -3 \\ 5 & -3 & 1 \end{bmatrix} \text{ is not symmetric}$$

Matrix of a Quadratic Form

Definition

Let $Q((x_1, \ldots, x_n)) = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{1 \leqslant i < j \leqslant n} a_{ij} x_i x_j$ be a quadratic form. The matrix of the quadratic form Q is a symmetric matrix $M = [b_{ij}] \in M(n \times n; \mathbb{R})$ such that $b_{ii} = a_{ii}$ and $b_{ij} = b_{ji} = \frac{1}{2} a_{ij}$ for $1 \leqslant i < j \leqslant n$.

Example

The matrix of the form $Q((x_1, x_2)) = x_1^2 - x_2^2$ is $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

The matrix of the form $Q((x_1, x_2, x_3)) = x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2$ $-4x_1x_3 + 8x_2x_3$ is

$$M = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 4 \\ -2 & 4 & 5 \end{bmatrix}$$

Matrix of a Quadratic Form (continued)

Proposition

Let M be a matrix of the quadratic form $Q\colon \mathbb{R}^n \longrightarrow \mathbb{R}$. Then

$$Q((x_1, \ldots, x_n)) = x^\intercal M x,$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

Proof.

Entries of matrix M in the i-th row get multiplied by x_i and elements in the j-th column get multiplied by x_j. For every $i \neq j$ the monomial $x_i x_j$ comes from the entry in the i-th row, j-th column and from from the entry in the j-th row, i-th column. $\square$

Positive/Negative Definite Quadratic Form

Definition

Quadratic form $Q\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ (resp. symmetric matrix $M \in M(n \times n; \mathbb{R})$) is **positive definite** if $Q(x) > 0$ (resp. $x^\intercal M x > 0$) for any $x \in \mathbb{R}^n,\ x \neq \mathbf{0}$.
Quadratic form Q (resp. symmetric matrix M) is **negative definite** if $Q(x) < 0$ (resp. $x^\intercal M x < 0$) for any $x \in \mathbb{R}^n,\ x \neq \mathbf{0}$.

Example

The quadratic form $\|\cdot\|^2\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is positive definite since $\|x\|^2 = x_1^2 + \ldots + x_n^2 > 0$ for $x \neq \mathbf{0}$.
The quadratic form $Q((x_1, x_2)) = x_1^2 - x_2^2$ is not positive definite since $Q((0,1)) = -1 < 0$. It is not negative definite since $Q((1,0)) = 1 > 0$.
The quadratic form $Q((x_1, x_2, x_3)) = x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3 = (x_1 + x_2 - x_3)^2 + (x_2 + 2x_3)^2$ is not positive definite since $Q((3, -2, 1)) = 0$. It is not negative definite.

Recall

$$(a_1+a_2+\ldots+a_n)^2 = a_1^2+a_2^2+\ldots+a_n^2+2a_1a_2+2a_1a_3+\ldots+2a_1a_n+$$

$$+2a_2a_3+2a_2a_4+\ldots+2a_2a_n+2a_3a_4+\ldots+2a_3a_n+\ldots\ldots+2a_{n-1}a_n$$

For example

$$(x_1 - 3x_2 + 2x_3)^2 =$$

$$= x_1^2+(-3)^2x_2^2+2^2x_3^2+2\cdot(-3)x_1x_2+2\cdot 2x_1x_3+2\cdot(-3)\cdot 2x_2x_3 =$$
$$= x_1^2 + 9x_2^2 + 4x_3^2 - 6x_1x_2 + 4x_1x_3 - 12x_2x_3$$

Proposition

A quadratic form $Q\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ can expressed (possibly after a change of coordinates) as $Q((x_1,\ldots,x_n)) = \pm l_1^2 \pm l_2^2 \pm \ldots \pm l_n^2$ where $l_1,\ldots,l_n$ are linear functions such that $l_i,\ldots,l_n$ do not depend on the variables $x_1,\ldots,x_{i-1}$ for $i = 2,\ldots,n$.

Proof.

(sketch) Use the above formula. □

Example

$$Q((x_1,x_2,x_3)) = x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 \; -2x_1x_3 + 2x_2x_3 =$$
$$(x_1+x_2-x_3)^2 + x_2^2 + 4x_2x_3 + 4x_3^2 = (x_1+x_2-x_3)^2 + (x_2+2x_3)^2$$

$$Q((x_1,x_2,x_3)) = x_1^2 - x_2^2 + x_3^2 + 2x_1x_2 - 4x_1x_3 =$$
$$(x_1+x_2-2x_3)^2 - 2x_2^2 + 4x_2x_3 - 3x_3^2 = (x_1+x_2-2x_3)^2 - 2(x_2-x_3)^2 - x_3^2$$

What to do if there is no square? Do the following substitution:

$$Q((x_1,x_2,x_3)) = x_1x_2 + 2x_1x_3 = \left\{\begin{smallmatrix} x_1=y_1-y_2 \\ x_2=y_1+y_2 \\ x_3=y_3 \end{smallmatrix}\right\} =$$
$$(y_1-y_2)(y_1+y_2) + 2(y_1-y_2)y_3 = y_1^2 - y_2^2 + 2y_1y_3 - 2y_2y_3 =$$
$$(y_1+y_3)^2 - y_2^2 - y_3^2 - 2y_2y_3 = (y_1+y_3)^2 - (y_2+y_3)^2$$

Sylvester's Criterion

Proposition

Let $M \in M(n \times n; \mathbb{R})$ be a symmetric matrix. Let W_i denote the determinant of the upper-left i-by-i submatrix of M. Matrix M is positive definite if and only if $W_i > 0$ for $i = 1, \ldots, n$.

The determinants W_i are sometimes called **leading principal minors**.

Example

Consider the symmetric matrix

$$M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 6 \end{bmatrix}$$

and compute its leading principal minors

$W_1 = \det \begin{bmatrix} 1 \end{bmatrix} = 1 > 0,$

$W_2 = \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1 > 0,$

$W_3 = \det \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 6 \end{bmatrix} \overset{\substack{c_1+c_3 \\ c_2+c_3}}{=} \det \begin{bmatrix} 0 & 0 & -1 \\ 2 & 3 & 1 \\ 5 & 7 & 6 \end{bmatrix} = 1 > 0.$

By Sylverster's criterion the quadratic form $x_1^2 + 2x_2^2 + 6x_3^2 + 2x_1x_2$ $-2x_1x_3 + 2x_2x_3$ is positive definite.

Another Example

Consider the symmetric matrix

$$M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 5 \end{bmatrix}$$

and compute its leading principal minors

$W_1 = \det \begin{bmatrix} 1 \end{bmatrix} = 1 > 0,$

$W_2 = \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1 > 0,$

$W_3 = \det \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 5 \end{bmatrix} \overset{\substack{c_1+c_3 \\ c_2+c_3}}{=} \det \begin{bmatrix} 0 & 0 & -1 \\ 2 & 3 & 1 \\ 4 & 6 & 5 \end{bmatrix} = 0 \not> 0.$

By Sylverster's criterion the quadratic form $Q((x_1, x_2, x_3)) = x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$ is not positive definite. In fact, $Q((3, -2, 1)) = 0$.

Sylvester's Criterion (continued)

A quadratic form Q is positive definite if and only if $-Q$ is negative definite.

Proposition

Let $M \in M(n \times n; \mathbb{R})$ be a symmetric matrix. Let W_i denote the determinant of the upper-left i-by-i submatrix of M. Matrix M is negative definite if and only if

$$W_i < 0 \text{ for odd } i,$$

$$W_i > 0 \text{ for even } i,$$

for $i = 1, \ldots, n$.

Example

Consider the symmetric matrix

$$M = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$$

and compute its leading principal minors
$W_1 = \det \begin{bmatrix} -1 \end{bmatrix} = -1 < 0$,
$W_2 = \det \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} = 1 > 0$,
The quadratic form $-x_1^2 - 2x_1x_2 - 2x_2^2 = -(x_1 + x_2)^2 - x_2^2$ is negative definite.

Postive/Negative Semidefinite Form

Definition

Quadratic form $Q\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ (resp. symmetric matrix $M \in M(n \times n; \mathbb{R})$) is **positive semidefinite** if $Q(x) \geqslant 0$ (resp. $x^\intercal M x \geqslant 0$) for any $x \in \mathbb{R}^n$.
Quadratic form Q (resp. symmetric matrix M) is **negative semidefinite** if $Q(x) \leqslant 0$ (resp. $x^\intercal M x \leqslant 0$) for any $x \in \mathbb{R}^n$.
Quadratic form Q (resp. symmetric matrix M) is **indefinite** if there exist $x, y \in \mathbb{R}^n$ such that $Q(x) > 0, Q(y) < 0$ (resp. $x^\intercal M x > 0, y^\intercal M y < 0$).

Remark

A positive (resp. negative) defined quadratic form is positive (resp. negative) semidefinite. A quadratic form is indefinite if and only if it is not positive semidefinite and it is not negative semidefinite.

Examples

The quadratic form $Q((x_1, x_2)) = x_1^2 - x_2^2$ is indefinite since $Q((1, 0)) > 0$ and $Q((0, 1)) < 0$.

The quadratic form $Q((x_1, x_2, x_3)) = x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2$ $-2x_1x_3 + 2x_2x_3 = (x_1 + x_2 - x_3)^2 + (x_2 + 2x_3)^2$ is positive semidefinite. It is not positive definite.

The quadratic form $Q((x_1, x_2)) = -x_1^2 - 2x_1x_2 - 2x_2^2 = -(x_1 + x_2)^2 - x_2^2$ is negative definite.

The quadratic form $Q((x_1, x_2, x_3)) = -x_1^2 - 2x_1x_2 - 2x_2^2 = -(x_1 + x_2)^2 - x_2^2$ is not negative definite since $Q((0, 0, 1)) = 0$. It is negative semidefinite.

Warning

Consider the quadratic form $Q((x_1, x_2)) = -x_2^2$. It is negative semidefinite. Its matrix is

$$M = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

Compute the leading principal minors
$W_1 = \det \begin{bmatrix} 0 \end{bmatrix} = 0 \geqslant 0,$
$W_2 = \det \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = 0 \geqslant 0,$
This shows **there is no** direct analogue of Sylvester's criterion for positive/negative semidefinite matrices.

Eigenvalues and Positivity

Theorem

Symmetric matrix $M \in M(n \times n; \mathbb{R})$ is diagonalizable.

In particular, the characteristic polynomial $w_M(\lambda) = \det(M - \lambda I)$ has n real roots (=eigenvalues) counted with multiplicities .

Theorem

Let $Q\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quadratic form and let M be its matrix. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be the roots of $w_M(\lambda)$. Then

i) *form Q is positive definite $\iff \lambda_1, \ldots, \lambda_n > 0$,*
ii) *form Q is positive semidefinite $\iff \lambda_1, \ldots, \lambda_n \geqslant 0$,*
iii) *form Q is negative definite $\iff \lambda_1, \ldots, \lambda_n < 0$,*
iv) *form Q is negative semidefinite $\iff \lambda_1, \ldots, \lambda_n \leqslant 0$,*
v) *form Q is indefinite $\iff \lambda_i < 0, \lambda_j > 0$ for some $1 \leqslant i, j \leqslant n$.*

Eigenvalues and Positivity (continued)

Proof.

Let $v_1, \dots, v_n \in \mathbb{R}^n$ be a basis of $\mathbb{R}^n$ consisting of eigenvectors of M, that is

$$Mv_i = \lambda_i v_i \text{ for } i = 1, \dots, n,$$

where $\lambda_i \in \mathbb{R}$ is an eigenvalue of M and $v_i = \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} \in M(n \times 1; \mathbb{R})$ is taken to be a n-by-1 matrix. Let v_i, v_j be vectors such that $\lambda_i \neq \lambda_j$. Then

$$v_i^\mathsf{T} M v_j = v_i^\mathsf{T}(Mv_j) = v_i^\mathsf{T}(\lambda_j v_j) = \lambda_j(v_i \cdot v_j),$$
$$v_i^\mathsf{T} M v_j = (v_i^\mathsf{T} M^\mathsf{T})v_j = (Mv_i)^\mathsf{T} v_j = (\lambda_i v_i)^\mathsf{T} v_j = \lambda_i(v_i \cdot v_j).$$

This is possible only if $v_i \cdot v_j = 0$, i.e. vectors v_i, v_j are perpendicular. Using Gram-Schmidt process for eigenspaces $V_{(\lambda_i)}$ one can assume the basis $v_1, \dots, v_n$ is orthonormal.

Eigenvalues and Positivity (continued)

Proof.

That is

$$v_i \cdot v_j = v_i^{\mathsf{T}} v_j = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{cases}.$$

For any $v \in \mathbb{R}^n$ there exist unique $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that

$$v = \alpha_1 v_1 + \ldots + \alpha_n v_n.$$

Now

$$Q(v) = v^{\mathsf{T}} M v = v^{\mathsf{T}} M(\alpha_1 v_1 + \ldots + \alpha_n v_n) =$$

$$= (\alpha_1 v_1 + \ldots + \alpha_n v_n)^{\mathsf{T}} (\lambda_1 \alpha_1 v_1 + \ldots + \lambda_n \alpha_n v_n) = \lambda_1 \alpha_1^2 + \ldots + \lambda_n \alpha_n^2.$$

In particular

$$Q(v_i) = v_i^{\mathsf{T}} M v_i = \lambda_i.$$

$\square$

Example

Let

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 1 > 0, \lambda_2 = -1 < 0$ therefore the quadratic form $Q((x_1, x_2)) = x_1^2 - x_2^2$ is indefinite.

Example

Let

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

The characteristic polynomial
$w_M(\lambda) = (1-\lambda)((2-\lambda)^2 - 4) = \lambda(1-\lambda)(\lambda-4)$ has
non-negative roots $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 4,\ \lambda_1, \lambda_2, \lambda_3 \geqslant 0$.
Therefore the quadratic form
$Q((x_1, x_2, x_3)) = x_1^2 + 2x_2^2 + 2x_3^2 + 4x_2x_3 = x_1^2 + 2(x_2 + x_3)^2$ is
positive semidefinite.