Linear Algebra

Lecture 6 - Determinants

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Notation

Definition

A matrix $A \in M(n \times n; \mathbb{R})$ is called a square matrix. For any square matrix A let $A_{ij} \in M((n-1) \times (n-1); \mathbb{R})$ denote the submatrix of A formed by deleting the i-th row and j-th column of A.

Example

$$A = \begin{bmatrix} -1 & 5 & 0 \\ 4 & -2 & 3 \\ 2 & -1 & 0 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -1 & 5 \\ 2 & -1 \end{bmatrix}.$$

Determinant

Definition

A determinant is a function $\det : M(n \times n; \mathbb{R}) \longrightarrow \mathbb{R}$ satisfying the conditions:

i) if $A = [a]$ then $\det A = a$,

ii) if $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$ and $n > 1$ then

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}.$$

Examples

In particular, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

$$\det A = (-1)^{1+1} a_{11} a_{22} + (-1)^{1+2} a_{12} a_{21} = a_{11} a_{22} - a_{12} a_{21}$$

.

For example, $\det \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = 1 \cdot 4 - 3 \cdot 2 = -2.$

Examples (continued)

In particular, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\det A = (-1)^{1+1} a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} +$$

$$(-1)^{1+2} a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + (-1)^{1+3} a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} =$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

For example, $\det \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 2 & 2 \end{bmatrix} = 1 \cdot 3 \cdot 2 + 2 \cdot 1 \cdot 2 = 10.$

Rule of Sarrus

$$\begin{array}{c}
\begin{array}{ccccc} + & + & + & \phantom{a_{11}} & \phantom{a_{12}} \end{array} \\
\left[\begin{array}{ccc|cc}
a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
a_{31} & a_{32} & a_{33} & a_{31} & a_{32}
\end{array}\right] \\
\begin{array}{ccccc} - & - & - & \phantom{a_{11}} & \phantom{a_{12}} \end{array}
\end{array}$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Note this DOES NOT work for *n*-by-*n* matrices for $n \geqslant 4$.

Properties of Determinants

Let $A, B, C \in M(n \times n; \mathbb{R})$

Theorem

i) *Let* $1 \leqslant k \leqslant n$*. If matrices* A, B, C *have all rows the same (resp. columns) except the k-th row (resp.column) and k-th row of* C *is the sum of k-th rows (resp. columns) of matrices* A *and* B *then* $\det C = \det A + \det B$*,*

ii) *If matrix* B *is equal to the matrix* A *with two rows (resp. columns) interchanged then* $\det B = -\det A$*,*

iii) *If matrix* B *is equal to the matrix* A *with some row (res. some column) multiplied by a constant* $c \in \mathbb{R}$ *then* $\det B = c \det A$*.*

Proof.

Use induction on the matrix size. $\square$

Examples

i)

$$\det\begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 2 & 2 \end{bmatrix} + \det\begin{bmatrix} 1 & 0 & 2 \\ 2 & -5 & 3 \\ 0 & 2 & 2 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 3 \\ 0 & 2 & 2 \end{bmatrix}$$

ii)

$$\det\begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix} = -\det\begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

iii)

$$\det\begin{bmatrix} 1 & 0 & 2 \\ 3 & 9 & 0 \\ 0 & 2 & 2 \end{bmatrix} = 3\det\begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

Transposition

Definition

Let $A = [a_{ij}] \in M(m \times n; \mathbb{R})$. The matrix $B = [b_{ij}] \in M(n \times m; \mathbb{R})$ where $b_{ij} = a_{ji}$ is called **the transpose** of matrix A. We write $B = A^\intercal$.

Example

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 5 \end{bmatrix}, \quad A^\intercal = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 0 & 5 \end{bmatrix}.$$

Theorem

Let $A \in M(n \times n; \mathbb{R})$. Then $\det A = \det A^\intercal$.

Example

$$\det \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}^\intercal = \det \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \det \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}.$$

Laplace expansion

Theorem

Let $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$ *and let* $n > 1$. *Then for any* $1 \leqslant i \leqslant n$

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ji} \det A_{ji}.$$

Example

$$\det \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 9 & 2 & 0 \\ 3 & 8 & 4 & 3 \\ 2 & 6 & 5 & 0 \end{bmatrix} = (-1)^{3+4} 3 \det \begin{bmatrix} 0 & 2 & 0 \\ 1 & 9 & 2 \\ 2 & 6 & 5 \end{bmatrix} =$$

$$= -3(-1)^{1+2} 2 \det \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = 6.$$

Determinants and Matrix Multiplication

Theorem (Special case of Cauchy-Binet formula)

Let $A, B \in M(n \times n; \mathbb{R})$*. Then* $\det AB = \det A \det B$*.*

Example

$$\det \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \det \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$$

Corollaries

Corollary

i) *if matrix A has a zero row or a zero column then* $\det A = 0$,

ii) *if matrix A has two equal rows (resp. columns) then* $\det A = 0$,

iii) *an elementary operation of switching two rows (resp. columns) of matrix A changes the sign of the determinant of A,*

iv) *an elementary operation of adding a row (resp. a column) of matrix A to other row (resp. column) does not change the determinant of A,*

v) *an elementary operation of multiplying a row (resp. a column) of matrix A by a constant $c \in \mathbb{R}$ multiplies the determinant by constant c,*

vi) *if rows (resp. columns) of matrix A form are linearly dependent then* $\det A = 0$.

Proofs

i) use Laplace expansion formula along the zero row (resp. column),

ii) use induction on the size of the matrix,

iii) as above,

iv) use Laplace expansion formula along the row (resp. column) which is the sum,

v) use Laplace expansion formula along the row (resp. column) multiplied by $c \in \mathbb{R}$,

vi) a row (resp. a column) is a linear combination of the other, use elementary row (resp. column) operations to get a zero row (resp. a zero column). Then use i).

Computing Determinants

Definition

A matrix $A = [a_{ij}] \in M(n \times n; \mathbb{R})$ is called upper-triangular if $a_{ij} = 0$ for $1 \leqslant j < i \leqslant n$.

Example

Matrix $\begin{bmatrix} 1 & 0 & 1 & -1 & 7 \\ 0 & 3 & 0 & 2 & 3 \\ 0 & 0 & 5 & 0 & -2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is upper-triangular.

Proposition

If matrix $A = [a_{ij}] \in M(n \times n; \mathbb{R})$ *is upper-triangular then* $\det A = a_{11} a_{22} \cdots a_{nn}$.

Proof.

Use induction and the Laplace expansion formula along the first column of A. $\square$

Computing Determinants (continued)

Note that a square matrix in an echelon form is upper-triangular.

Corollary

For any $A \in M(n \times n; \mathbb{R})$ *rows (resp. columns) of* A *are linearly dependent if and only if* $\det A = 0$.

Proof.

$(\Leftarrow)$ matrix A can be transformed by elementary row operations to an echelon form with a zero row. $\square$

How to compute determinant of matrix?

Use elementary operations on rows and columns in order to get as many zeroes as possible in a row or a column and use the Laplace expansion.
or
Put matrix in an upper-triangular form using elementary operations and take product of the diagonal entries.

Example

$$\det\begin{bmatrix} 1 & 2 & 2 & 6 \\ 1 & 2 & 2 & 5 \\ 1 & 1 & 2 & 8 \\ 2 & 5 & 6 & 2 \end{bmatrix} \overset{r_1 - r_2}{=} \det\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & 2 & 5 \\ 1 & 1 & 2 & 8 \\ 2 & 5 & 6 & 2 \end{bmatrix} =$$

$$(-1)^{1+4}\det\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 5 & 6 \end{bmatrix} = -2\det\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 5 & 3 \end{bmatrix} \overset{c_3 - c_1}{=}$$

$$-2\det\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} = -2(-1)^{3+3}\det\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = 2.$$

Block Matrices

Theorem

Let $M = \begin{bmatrix} A & B \\ \mathbf{0} & C \end{bmatrix}$ *where* A, C *are square matrices and* $\mathbf{0}$ *is a zero matrix. Then* $\det M = \det A \det C$.

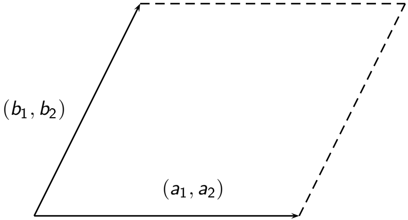
Example

Let

$$\det \begin{bmatrix} 1 & 2 & 1 & -1 & 3 \\ 3 & 0 & 2 & 10 & 22 \\ 4 & 5 & 0 & 7 & 9 \\ 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix} \det \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} =$$

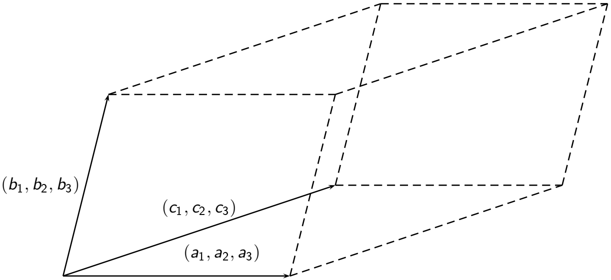
$(2 \cdot 2 \cdot 4 + 1 \cdot 3 \cdot 5 - 1 \cdot 2 \cdot 5)(2 \cdot 2 - 1 \cdot 5) = -21.$

Area (2−dimensional volume)



The area of a parallelogram spanned by vectors $(a_1, a_2), (b_1, b_2)$ is equal to the absolute value of $\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

Volume



The area of a parallelepiped spanned by vectors $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)$ is equal to the absolute value of

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$