

Linear Algebra

Lecture 4 - Linear Transformations

Oskar Kędzierski

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Linear Transformations

Definition

Let V, W be vector spaces. A function $\varphi : V \longrightarrow W$ is a **linear transformation** if

- i) $\varphi(v + w) = \varphi(v) + \varphi(w)$ for all $v, w \in V$,
- ii) $\varphi(\alpha v) = \alpha\varphi(v)$ for all $\alpha \in \mathbb{R}$ and $v \in V$.

For example, the function $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $\varphi((x_1, x_2, x_3)) = (x_1 + 2x_2 - x_3, x_2 + x_3)$ is a linear transformation.

In general, the following theorem holds:

Linear Transformations (continued)

Theorem

A function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if and only if

$$\varphi((x_1, \dots, x_n)) = (a_{11}x_1 + \dots + a_{1n}x_n, a_{21}x_1 + \dots + a_{2n}x_n, \dots \\ \dots, a_{m1}x_1 + \dots + a_{mn}x_n), \text{ where } a_{ij} \in \mathbb{R}.$$

Proof.

Assume φ is a linear transformation. Let

$$\varphi(\varepsilon_i) = (a_{1i}, a_{2i}, \dots, a_{mi}).$$

Then, by linearity,

$$\varphi(x_1, \dots, x_n) = \varphi(x_1\varepsilon_1 + \dots + x_n\varepsilon_n) = x_1\varphi(\varepsilon_1) + \dots + x_n\varphi(\varepsilon_n) = \\ (a_{11}x_1 + \dots + a_{1n}x_n, a_{21}x_1 + \dots + a_{2n}x_n, \dots, a_{m1}x_1 + \dots + a_{mn}x_n). \quad \square$$

Examples

The transformation $\varphi : \mathcal{F}(\mathbb{R}, \mathbb{R}) \longrightarrow \mathbb{R}$ defined by $\varphi(f) = f(1)$ is linear.

The transformation $\varphi : \mathcal{D}(\mathbb{R}, \mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R}, \mathbb{R})$, where $\mathcal{D}(\mathbb{R}, \mathbb{R})$ denotes differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $\varphi(f) = f'$ is linear.

Properties

Let $\varphi : V \longrightarrow W$ be a linear transformation. Then

- i) $\varphi(\mathbf{0}) = \mathbf{0}$,
- ii) if vectors v_1, \dots, v_k are linearly dependent then $\varphi(v_1), \dots, \varphi(v_k)$ are linearly dependent too,
- iii) the set $\varphi(V) = \{\varphi(v) \mid v \in V\}$ is a subspace of W called the **image** of V , if v_1, \dots, v_k span V then $\varphi(v_1), \dots, \varphi(v_k)$ span $\varphi(V)$,
- iv) the transformation φ is injective if and only if $\{v \in V \mid \varphi(v) = \mathbf{0}\} = \{\mathbf{0}\}$,
- v) if φ is injective and the vectors v_1, \dots, v_k are linearly independent then $\varphi(v_1), \dots, \varphi(v_k)$ are linearly independent too.

Properties (continued)

Theorem

Let V, W be vector spaces. For any basis $v_1, \dots, v_n \in V$ and any vectors $w_1, \dots, w_n \in W$ there exists a unique linear transformation $\varphi : V \longrightarrow W$ such that $\varphi(v_i) = w_i$ for $i = 1, \dots, n$.

Proof.

For $v = \sum_{i=1}^n \alpha_i v_i$ set $\varphi(v) = \sum_{i=1}^n \alpha_i w_i$. It is easy to check that φ is a linear transformation (by the uniqueness of coordinates relative to a basis) and it is unique, since any other linear transformation sending v_i to w_i satisfies the same conditions. □

Example

Let $\varphi; \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation such that $\varphi((1, 3)) = (1, 1, 1)$ and $\varphi((0, 1)) = (-1, 0, 2)$.

Then $\varphi((1, 0)) = \varphi((1, 3) - 3(0, 1)) = \varphi((1, 3)) - 3\varphi((0, 1)) = (1, 1, 1) - 3(-1, 0, 2) = (4, 1, -5)$.

Therefore, $\varphi((x_1, x_2)) = \varphi(x_1\varepsilon_1 + x_2\varepsilon_2) = x_1(4, 1, -5) + x_2(-1, 0, 2) = (4x_1 - x_2, x_1, -5x_1 + 2x_2)$.

Representation of Transformation by Matrices

Recall that by $M(m \times n; \mathbb{R})$ we denote the set of real matrices with m rows and n columns.

Definition

Let V, W be vector spaces and let

$\mathcal{A} = (v_1, \dots, v_n), \mathcal{B} = (w_1, \dots, w_m)$ be their ordered bases, respectively. The **matrix of a linear transformation** $\varphi: V \longrightarrow W$ relative to the pair of ordered bases \mathcal{A} and \mathcal{B} is matrix

$M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = [a_{ij}] \in M(m \times n; \mathbb{R})$ given by the conditions $\varphi(v_j) = \sum_{i=1}^m a_{ij} w_i$ for $j = 1, \dots, n$.

That is, columns of $M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$ consist of coefficients of the vectors $\varphi(v_1), \dots, \varphi(v_n)$ relative to the basis \mathcal{B} .

Examples

Let $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation defined by $\varphi((x_1, x_2, x_3)) = (2x_1 - x_2 + x_3, x_1 + x_3)$ and let $\mathcal{A} = ((1, 0, 1), (2, 0, 3), (0, 1, 1))$, $\mathcal{B} = ((1, 1), (0, 1))$ be the ordered bases of \mathbb{R}^3 and \mathbb{R}^2 , respectively. Then

$$\varphi((1, 0, 1)) = (3, 2) = 3(1, 1) - 1(0, 1)$$

$$\varphi((2, 0, 3)) = (7, 5) = 7(1, 1) - 2(0, 1)$$

$$\varphi((0, 1, 1)) = (0, 1) = 0(1, 1) + 1(0, 1).$$

The matrix of φ relative to the (ordered) bases \mathcal{A}, \mathcal{B} is

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 3 & 7 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

Examples (continued)

Notation

By st we will denote the standard basis of \mathbb{R}^n , i.e.

$$st = (\varepsilon_1, \dots, \varepsilon_n).$$

For example, the matrix of the linear transformation given by $\varphi((x_1, x_2, x_3)) = (2x_1 - x_2 + 3x_3, x_1 + x_3)$ relative to the standard bases in \mathbb{R}^3 and \mathbb{R}^2 is

$$M(\varphi)_{st}^{st} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

since

$$\varphi(\varepsilon_1) = \varphi((1, 0, 0)) = (2, 1) = 2(1, 0) + 1(0, 1) = 2\varepsilon_1 + 1\varepsilon_2$$

$$\varphi(\varepsilon_2) = \varphi((0, 1, 0)) = (-1, 0) = -1(1, 0) + 0(0, 1) = -1\varepsilon_1 + 0\varepsilon_2$$

$$\varphi(\varepsilon_3) = \varphi((0, 0, 1)) = (3, 1) = 3(1, 0) + 1(0, 1) = 3\varepsilon_1 + 1\varepsilon_2.$$

Elementary operations and matrices of linear transformations

Proposition

Let V, W be vector spaces and let $\mathcal{A} = (v_1, \dots, v_n)$, $\mathcal{B} = (w_1, \dots, w_m)$ be their ordered bases, respectively. Let $A = [a_{ij}] = M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$ be the matrix of a linear transformation $\varphi: V \longrightarrow W$ relative to the bases \mathcal{A} and \mathcal{B} . If $\mathcal{A}' = (v_1 + v_2, v_2, \dots, v_n)$, $\mathcal{A}'' = (\alpha v_1, v_2, \dots, v_n)$, $\mathcal{B}' = (w_1 + w_2, w_2, \dots, w_m)$, $\mathcal{B}'' = (\alpha w_1, w_2, \dots, w_m)$ for some $\alpha \neq 0$ then

$$M(\varphi)_{\mathcal{A}'}^{\mathcal{B}'} = \begin{bmatrix} a_{11} + a_{12} & a_{12} & \dots & a_{1n} \\ a_{21} + a_{22} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + a_{m2} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

Elementary operations and matrices of linear transformations (continued)

$$M(\varphi)_{\mathcal{A}''}^{\mathcal{B}} = \begin{bmatrix} \alpha a_{11} & a_{12} & \dots & a_{1n} \\ \alpha a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}'} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} - a_{11} & a_{22} - a_{12} & \dots & a_{2n} - a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}''} = \begin{bmatrix} a_{11}/\alpha & a_{12}/\alpha & \dots & a_{1n}/\alpha \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Rank

Definition

Let $k_1, \dots, k_n \in \mathbb{R}^m$ denote the vectors corresponding to columns of matrix $A \in M(m \times n; \mathbb{R})$. The dimension of the space $\text{lin}(k_1, \dots, k_n) \subset \mathbb{R}^m$ will be called **rank** of the matrix A and denoted by $r(A)$.

Proposition

Let $\varphi : V \longrightarrow W$ be a linear transformation and let \mathcal{A}, \mathcal{B} be ordered bases of V and W , respectively. Then the rank of the matrix $M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$ is equal to $\dim \varphi(V)$ and hence it does not depend on the bases \mathcal{A}, \mathcal{B} .

Corollary

Elementary row operations on matrix A do not change its rank. Therefore, the rank of matrix A is equal to the rank of its (reduced) echelon form A' , which is equal to the number of non-zero rows in A' .