Linear Algebra

Lecture 8 - Linear Endomorphisms

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Endomorphism

Definition

Let V be a vector space and $\mathcal{A}$ its (ordered) basis. A linear transformation $\varphi : V \longrightarrow V$ is called a **linear endomorphism**. The matrix $M(\varphi)_{\mathcal{A}}^{\mathcal{A}}$ is called matrix of endomorphism relative to basis $\mathcal{A}$. It is denoted in short $M(\varphi)_{\mathcal{A}}$.

Example

The identity $id : V \longrightarrow V$ is a linear endomorphism and its matrix relative to any basis $\mathcal{A}$ is the identity matrix

$$M(id)_{\mathcal{A}} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in M(n \times n; \mathbb{R}),$$

where $n = \dim V$.

Example

Let

$$s : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

$$r : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

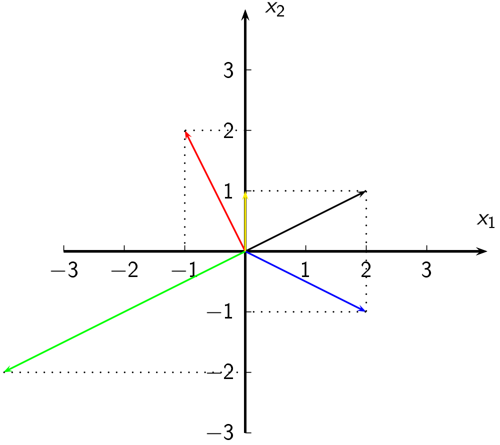
$$k : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

$$p : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

be linear endomorphisms of $\mathbb{R}^2$ defined as follows: s is a reflection of $\mathbb{R}^2$ about the x_1-axis, r rotation about the origin of $\mathbb{R}^2$ (i.e. $(0,0)$) by $\frac{\pi}{2}$ radians (i.e. 90 degrees) counter-clockwise, k is scaling by -2 in all directions (also called uniform scaling) and p is projection onto the x_2-axis.

Example (continued)

For example, if $v = (2, 1)$ then
$s(v) = (2, -1)$, $r(v) = (-1, 2)$, $k(v) = (-4, -2)$, $p(v) = (0, 1)$.



$s(x_1, x_2) = (x_1, -x_2)$

$r(x_1, x_2) = (-x_2, x_1)$

$k(x_1, x_2) = (-2x_1, -2x_2)$

$p(x_1, x_2) = (0, x_2)$

Example (continued)

$$s(x_1,x_2)=(x_1,-x_2),\ \ r(x_1,x_2)=(-x_2,x_1),$$

$$k(x_1,x_2)=(-2x_1,-2x_2),\ \ p(x_1,x_2)=(0,x_2).$$

The matrices of these endomorphisms relative to the standard basis $st=((1,0),(0,1))$ look as follows:

$$M(s)_{st}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ M(r)_{st}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},$$

$$M(k)_{st}=\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix},\ M(p)_{st}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Take different basis, for example $\mathcal{A}=((1,2)(1,1))$. The change-of-coordinate matrix is

$$M(id)_{st}^{\mathcal{A}}=(M(id)_{\mathcal{A}}^{st})^{-1}=\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}^{-1}=\begin{bmatrix}-1 & 1\\ 2 & -1\end{bmatrix}.$$

Example (continued)

Recall, $\mathcal{A} = ((1,2)(1,1))$ and $M(id)^{\mathcal{A}}_{st} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$.

$$\begin{aligned}
s(1,2) &= (1,-2) &&= -3(1,2) + 4(1,1),\\
s(1,1) &= (1,-1) &&= -2(1,2) + 3(1,1),\\
r(1,2) &= (-2,1) &&= \ \ 3(1,2) - 5(1,1),\\
r(1,1) &= (-1,1) &&= \ \ 2(1,2) - 3(1,1),\\
k(1,2) &= (-2,-4) &&= -2(1,2) + 0(1,1),\\
k(1,1) &= (-2,-2) &&= \ \ 0(1,2) - 2(1,1),\\
p(1,2) &= (0,2) &&= \ \ 2(1,2) - 2(1,1),\\
p(1,1) &= (0,1) &&= \ \ 1(1,2) - 1(1,1).
\end{aligned}$$

$$M(s)_{\mathcal{A}} = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}, \quad M(r)_{\mathcal{A}} = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix},$$

$$M(k)_{\mathcal{A}} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad M(p)_{\mathcal{A}} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}.$$

Example (continued)

We see that matrices of simple linear transformations look ‘nice’ relative to some bases and ‘not-that-nice’ relative to the others. That aim of this lecture if to find a way of computing those ‘nice’ ones in the general case. Note that determinants and the ranks of corresponding matrices did not change.

Matrix Similarity

Definition

Two matrices $A, B \in M(n \times n; \mathbb{R})$ are called **similar** if there exists an invertible matrix $C \in M(n \times n; \mathbb{R})$ such that

$$A = C^{-1}BC.$$

Proposition

Let $\varphi : V \longrightarrow V$ be a linear endomorphism of a finite dimensional vector space V. For any two bases $\mathcal{A}, \mathcal{B}$ of V the matrices $M(\varphi)_{\mathcal{A}}$ and $M(\varphi)_{\mathcal{B}}$ are similar.

Proof.

$$M(\varphi)_{\mathcal{B}}^{\mathcal{B}} = M(id \circ \varphi \circ \mathrm{id})_{\mathcal{B}}^{\mathcal{B}} = M(id)_{\mathcal{A}}^{\mathcal{B}} M(\varphi)_{\mathcal{A}}^{\mathcal{A}} M(id)_{\mathcal{B}}^{\mathcal{A}}.$$

Therefore

$$M(\varphi)_{\mathcal{B}} = C^{-1} M(\varphi)_{\mathcal{A}} C,$$

where $C = M(id)_{\mathcal{B}}^{\mathcal{A}}$. $\square$

Example

Let $\varphi((x_1, x_2)) = (x_1 + x_2, 2x_1 + 3x_2)$ be a linear endomorphism $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Take $\mathcal{A} = st$ and $\mathcal{B} = ((-2,1),(1,-1)$. Then

$$M(\varphi)_{\mathcal{A}} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \text{ and } C = M(id)_{\mathcal{B}}^{\mathcal{A}} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Use $M(\varphi)_{\mathcal{B}} = C^{-1} M(\varphi)_{\mathcal{A}} C$ and compute $C^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$.

Then

$$M(\varphi)_{\mathcal{B}} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}.$$

On the other hand,

$$\varphi((-2,1)) = (-1,-1) = 2(-2,1) + 3(1,-1),$$

$$\varphi((1,-1)) = (0,-1) = (-2,1) + 2(1,-1).$$

Similar Matrices and Endomorphisms

Theorem

Let V be n-dimensional vector space and let $A, B \in M(n \times n; \mathbb{R})$. Then

$$A, B \text{ are similar} \Longleftrightarrow \text{there exists an endomorphism } \varphi : V \longrightarrow V$$
$$\text{and bases } \mathcal{A}, \mathcal{B} \text{ of } V \text{ such that } M(\varphi)_{\mathcal{A}} = A \text{ and } M(\varphi)_{\mathcal{B}} = B.$$

Proof.

($\Leftarrow$) was done before.
($\Rightarrow$) there exits an invertible matrix $C \in M(n \times n; \mathbb{R})$ such that $B = C^{-1}AC$. Let $\mathcal{A}$ be any basis of the vector space V and let φ be the unique linear endomorphism given by the condition $M(\varphi)^{\mathcal{A}}_{\mathcal{A}} = A$. If $\mathcal{B}$ is given by the condition $C = M(id)^{\mathcal{A}}_{\mathcal{B}}$ then $B = M(\varphi)_{\mathcal{B}}$. $\square$

Eigenvalues and Eigenvectors

Definition

Let $\varphi : V \longrightarrow V$ be a linear endomorphism of a finite dimensional vector space V. A constant $\lambda \in \mathbb{R}$ is called **eigenvalue** of φ if there exists a non-zero vector $v \in V$ such that

$$\varphi(v) = \lambda v.$$

Such vector v is called an **eigenvector** of φ associated to the eigenvalue λ.

Remark (geometric interpretation)

A vector $v \in V$ is an eigenvector of φ if and only if $\varphi(\mathrm{lin}(v)) \subset \mathrm{lin}(v)$ and $\mathrm{lin}(v) \neq \{\mathbf{0}\}$, i.e. v is a non-zero vector and the line spanned by v is mapped into itself.

Eigenvalues and Eigenvectors (continued)

Let $\varphi : V \longrightarrow V$ be a linear endomorphism. For any eigenvalue λ of φ let $V_{(\lambda)}$ denote the set of all eigenvectors associated to λ together with the zero vector, i.e.

$$V_{(\lambda)} = \{v \in V \mid \varphi(v) = \lambda v.\}$$

Proposition

The subset $V_{(\lambda)} \subset V$ is a subspace of V.

Proof.

Let $v, w \in V_{(\lambda)}$. Then $\varphi(v + w) = \varphi(v) + \varphi(w) = \lambda v + \lambda w = \lambda(v + w)$. Hence $v + w \in V_{(\lambda)}$. For any $\alpha \in \mathbb{R}$ we have $\varphi(\alpha v) = \alpha\varphi(v) = \lambda(\alpha v)$. Hence $\alpha v \in V_{(\lambda)}$. $\square$

For any eigenvalue λ of φ the subspace $V_{(\lambda)}$ is called **the eigenspace** associated to λ. It is straightforward that $\varphi(V_{(\lambda)}) \subset V_{(\lambda)}$.

Example

Let $s : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a reflection of $\mathbb{R}^2$ about the x_1-axis. Then $V_{(1)} = \mathrm{lin}((1,0))$ and $V_{(-1)} = \mathrm{lin}((0,1))$. The rotation r about the origin of $\mathbb{R}^2$ by $\frac{\pi}{2}$ radians counter-clockwise has no eigenvalues (no line is mapped into itself). In the case of uniform scaling k by -2 in all directions any non-zero vector is eigenvector associated to -2, i.e. $V_{(-2)} = \mathbb{R}^2$.The projection p onto the x_2-axis has two eigenspaces: $V_{(0)} = \mathrm{lin}((1,0))$ and $V_{(1)} = \mathrm{lin}((0,1))$.
Note that for s, k and p there exist a basis (the standard one) consisting of eigenvectors. The matrices of those endomorphisms in the standard basis are diagonal.

$$M(s)_{st} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; M(k)_{st} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

$$M(p)_{st} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Characteristic Polynomial

Definition

Let $A \in M(n \times n; \mathbb{R})$. The polynomial $w_A(\lambda) = \det(A - \lambda I_n)$ is called the characteristic polynomial of A.

The degree of $w_A(\lambda)$ is equal to n.

Example

Let $A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$. Then

$$w_A(\lambda) = \det \begin{bmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda) - 6 = \lambda^2 - 7\lambda + 6.$$

Proposition

Let $A, B \in M(n \times n; \mathbb{R})$ be similar matrices. Then $w_A = w_B$.

Proof.

There exists an invertible matrix C such that $A = C^{-1}BC$. But $w_A(\lambda) = \det(A - \lambda I_n) = \det(C^{-1}BC - C^{-1}\lambda I_n C) = \det(C^{-1}(B - \lambda I_n)C) = (\det C)^{-1} \det(B - \lambda I_n) \det C = w_B(\lambda)$. $\square$

Characteristic Polynomial (continued)

Definition

Let $\varphi : V \longrightarrow V$ be a linear endomorphism of a finite dimensional vector space V. The characteristic polynomial w_φ of φ is the characteristic polynomial of matrix $M(\varphi)_\mathcal{A}$ where $\mathcal{A}$ is a basis of V.

By the previous proposition the characteristic polynomial of φ does not depend on the basis $\mathcal{A}$.

Finding Eigenvalues and Eigenvectors

Theorem

Let $\varphi : V \longrightarrow V$ be a linear endomorphism of a finite dimensional vector space V.

i) *$\alpha \in \mathbb{R}$ is an eigenvalue of $\varphi \iff \alpha$ is a root the characteristic polynomial of φ,*

ii) *let $\mathcal{A} = (v_1, \ldots, v_n)$ and $A = M(\varphi)_{\mathcal{A}}$. The vector $v = x_1 v_1 + \ldots + x_n v_n$ is an eigenvector of φ associated to α if and only if*

$$(A - \alpha I_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Finding Eigenvalues and Eigenvectors (continued)

Proof.

Let $v = x_1 v_1 + \ldots + x_n v_n$. Then $\varphi(v) = \alpha v$ if and only if

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \iff (A - \alpha I_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

From the previous lecture we know that there exists a non-zero solution of the latter if and only if $\det(A - \alpha I_n) = 0$, i.e. $w_A(\alpha) = 0$.

$\square$

Example

Let $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be an endomorphism of $\mathbb{R}^3$ given by $\varphi(x_1, x_2, x_3) = (4x_1 + 4x_2, -x_1, x_1 + 3x_2 + 3x_3)$. Its matrix in the standard basis is $A = M(\varphi)_{st} = \begin{bmatrix} 4 & 4 & 0 \\ -1 & 0 & 0 \\ 1 & 3 & 3 \end{bmatrix}$.

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 4 & 0 \\ -1 & -\lambda & 0 \\ 1 & 3 & 3-\lambda \end{bmatrix}.$$

Hence $w_\varphi(\lambda) = \det(A - \lambda I) = (3-\lambda)((4-\lambda)(-\lambda)+4) = (3-\lambda)(\lambda^2 - 4\lambda + 4) = (3-\lambda)(2-\lambda)^2$. There are two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$. To find $V_{(2)}$ we solve a system of linear equations:

$$V_{(2)} : \begin{bmatrix} 2 & 4 & 0 \\ -1 & -2 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Example (continued)

$$\begin{bmatrix} 2 & 4 & 0 \\ -1 & -2 & 0 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[r_3+r_2]{r_1+2r_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1-2r_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $x_1 = 2x_3,\ x_2 = -x_3,\ x_3 \in \mathbb{R}$, i.e.

$$V_{(2)} = \{(2x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}\} = \operatorname{lin}((2, -1, 1)).$$

$$V_{(3)} : \begin{bmatrix} 1 & 4 & 0 \\ -1 & -3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Example (continued)

$$\begin{bmatrix} 1 & 4 & 0 \\ -1 & -3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \xrightarrow[]{\substack{r_1+r_2 \\ r_3+r_2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_2+3r_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $x_1 = x_2 = 0,\ x_3 \in \mathbb{R}$, i.e.

$$V_{(3)} = \{(0, 0, x_3) \mid x_3 \in \mathbb{R}\} = \mathrm{lin}((0, 0, 1)).$$

Example (continued)

Recall that

$$\varphi(x_1, x_2, x_3) = (4x_1 + 4x_2, -x_1, x_1 + 3x_2 + 3x_3),$$

$$V_{(2)} = \mathrm{lin}((2, -1, 1)),$$

$$V_{(3)} = \mathrm{lin}((0, 0, 1)),$$

and check those directly

$$\varphi(2, -1, 1) = (4, -2, 2) = 2(2, -1, 1),$$

$$\varphi(0, 0, 1) = (0, 0, 3) = 3(0, 0, 1).$$

Remarks

i) if $\varphi : V \longrightarrow V$ and $\dim V$ is odd then the degree of w_φ is odd therefore it has at least one real root so there exists an eigenvector of φ,

ii) $\dim V_{(\alpha)} \leqslant$ multiplicity of the root α in w_φ, cf. the last example (2 is a root of multiplicity 2 but $\dim V_{(2)} = 1$),

iii) if $A \in M(n \times n; \mathbb{R})$ then $w_A(A) = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$, i.e. matrix A substituted to its characteristic polynomial gives the zero matrix (Cayley-Hamilton theorem).

Example

Let $A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ and $w_A(\lambda) = \lambda^2 - 2\lambda - 2$. Then

$$w_A(A) = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}^2 - 2\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & -6 \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$