Linear Algebra

Lecture 13 - Simplex Method

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Simplex Method

Simplex method is an algorithm solving linear programming problems presented in a standard form. It was invented by George Dantzig in 1947.

A linear programming problem in a standard form is a task of minimizing the objective function

$$f((x_1, \ldots, x_n)) = c_1 x_1 + \ldots + c_n x_n \longrightarrow min$$

under the constraints

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = b_m \end{cases}, \quad x_1, \ldots, x_n \geqslant 0$$

Simplex Method

Set

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

We assume that $r(A) = m$.
Let $X \subset \mathbb{R}^n$ be a convex polytope defined by the conditions $Ax = b, \; x \geqslant 0$. Recall, the if there is an optimal solution solution to the problem (i.e. a point $\overline{x} \in X$ in which f admits its minimum over X) then it can be chosen to be a vertex of X.

Vertices of X correspond to basic feasible solutions of the problem. They are given by basic feasible sets $\mathcal{B} \subset \{1, \ldots, n\}$ of $m = r(A)$ elements, such that the system of linear equations $Ax = b, \; x_i = 0$ for $i \notin \mathcal{B}$ has a unique non-negative solution.

Simplex Method

Simplex method starts from a basic feasible solutions. Then one moves to another basic feasible solution by replacing one element in the basic set $\mathcal{B}$ in order to decrease the value of the objective function f.

Example

Maximize the value $x_1 + 2x_2$ under the constraints

$$\begin{cases} x_1 & \geqslant & 0 & & \\ x_2 & \geqslant & 0 & & \\ 2x_1 & + & x_2 & \leqslant & 8 \\ -x_1 & + & 3x_2 & \leqslant & 3 \end{cases}$$

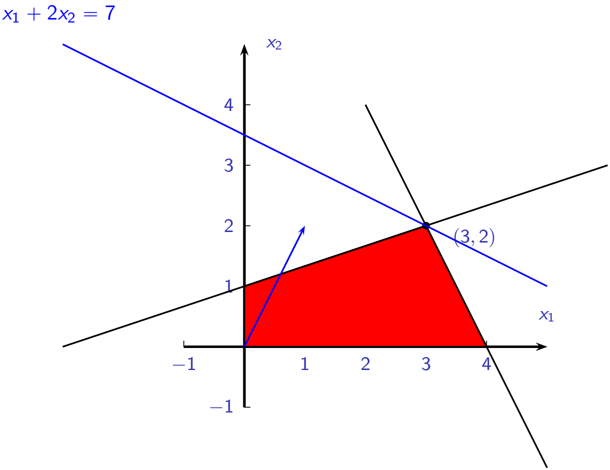
Express this problem in a standard form

$$-x_1 - 2x_2 \longrightarrow min$$

$$\begin{cases} 2x_1 & + & x_2 & + & x_3 & & & = & 8 \\ -x_1 & + & 3x_2 & & & + & x_4 & = & 3 \end{cases}$$

and $x_1, x_2, x_3, x_4 \geqslant 0$.

Example



maximize $x_1 + 2x_2$

$$\begin{cases} x_1 & \geqslant & 0 \\ x_2 & \geqslant & 0 \\ 2x_1 & + & x_2 & \leqslant & 8 \\ -x_1 & + & 3x_2 & \leqslant & 3 \end{cases}$$

optimal solution is $(3, 2)$

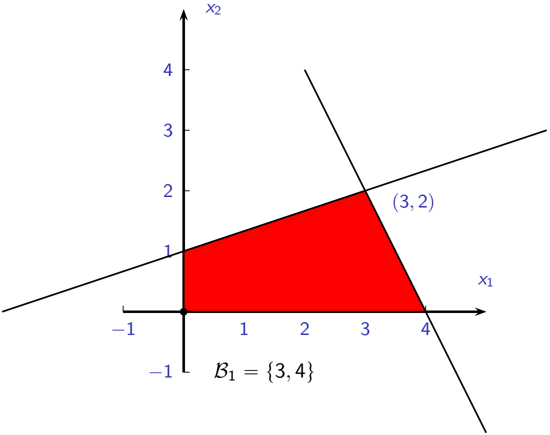
Example

We start from the basic feasible set $\mathcal{B}_1 = \{3, 4\}$. The basic variables are x_3, x_4 and the non-basic ones are x_1, x_2. The feasible basic solution is $\overline{x}_{\mathcal{B}_1} = (0, 0, 8, 3)$ which be computed directly from

$$\begin{cases} 2x_1 & + & x_2 & + & x_3 & & & = & 8 \\ -x_1 & + & 3x_2 & & & + & x_4 & = & 3 \end{cases}$$

by setting $x_1 = x_2 = 0$.
Since $f(x) = -x_1 - 2x_2$ therefore $f(\overline{x}_{\mathcal{B}_1}) = 0$. We could decrease it by making either x_1 or x_2 non-zero. By a heuristic rule we choose x_2 since the coefficient -2 is smaller than -1. Assume $s = 2$ will enter the new basic (feasible) set $\mathcal{B}_2$.

Example



maximize $x_1 + 2x_2$

$$\begin{cases} x_1 & \geqslant & 0 & & \\ x_2 & \geqslant & 0 & & \\ 2x_1 & + & x_2 & \leqslant & 8 \\ -x_1 & + & 3x_2 & \leqslant & 3 \end{cases}$$

Example

Since $s = 2$ enters the basic set we need to decide whether 3 or 4 leaves.

$$\begin{cases} 2x_1 + x_2 + x_3 = 8 \\ -x_1 + 3x_2 + x_4 = 3 \end{cases}$$

Divide the second equation by 3 to get coefficient at x_2 equal to 1

$$\begin{cases} 2x_1 + x_2 + x_3 = 8 \\ -\frac{1}{3}x_1 + x_2 + \frac{1}{3}x_4 = 1 \end{cases}$$

Subtract the first equation from the second to make x_2, x_4 basic variables. This means 3 leaves the basic set $\mathcal{B}_1$.

$$\begin{cases} 2x_1 + x_2 + x_3 = 8 \\ -\frac{7}{3}x_1 - x_3 + \frac{1}{3}x_4 = -7 \end{cases}$$

and $\overline{x}_{\{2,4\}} = (0, 8, 0, -21)$.

Example

$$\begin{cases} 2x_1 & + & x_2 & + & x_3 & & & = & 8 \\ -\frac{1}{3}x_1 & + & x_2 & & & + & \frac{1}{3}x_4 & = & 1 \end{cases}$$

Subtract the second equation from the first one to make x_2, x_3 basic variables. This means 4 leaves the basic set $\mathcal{B}_1$.

$$\begin{cases} \frac{7}{3}x_1 & & & + & x_3 & - & \frac{1}{3}x_4 & = & 7 \\ -\frac{1}{3}x_1 & + & x_2 & & & + & \frac{1}{3}x_4 & = & 1 \end{cases}$$

$$\text{and} \quad \overline{x}_{\{2,3\}} = (0, 1, 7, 0).$$

Both sets $\{2, 3\}$ and $\{2, 4\}$ are basic but only $\{2, 3\}$ is feasible since $\overline{x}_{\{2,3\}} = (0, 1, 7, 0) \geqslant 0$ and $\overline{x}_{\{2,4\}} = (0, 8, 0, -21) \ngeqslant 0$

Recall

$$\begin{cases} 2x_1 & + & 1x_2 & + & x_3 & & & = & 8 \\ -x_1 & + & 3x_2 & & & + & x_4 & = & 3 \end{cases}$$

Observe $\frac{8}{1} \geqslant \frac{3}{3}$. The crucial point is to subtract **smaller** ratio from the bigger one to get a positive number.

For $\mathcal{B}_2 = \{2, 3\}$ the general solution with x_2, x_3 as basic variables is

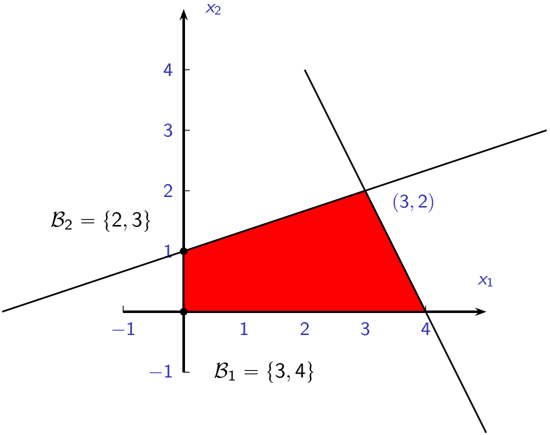
$$\begin{cases} -\frac{1}{3}x_1 & + & x_2 & & & + & \frac{1}{3}x_4 & = & 1 \\ \frac{7}{3}x_1 & & & + & x_3 & - & \frac{1}{3}x_4 & = & 7 \end{cases}$$

Substitute $x_2 = 1 + \frac{1}{3}x_1 - \frac{1}{3}x_4$ to $f(x)$

$$f(x) = -x_1 - 2x_2 = -2 - \frac{5}{3}x_1 + \frac{2}{3}x_4.$$

Making x_1 non-zero will decrease f, i.e. $s = 1$ will enter the new basic set $\mathcal{B}_3$.

Example



maximize $x_1 + 2x_2$

$$\begin{cases} x_1 & \geqslant & 0 & & \\ x_2 & \geqslant & 0 & & \\ 2x_1 & + & x_2 & \leqslant & 8 \\ -x_1 & + & 3x_2 & \leqslant & 3 \end{cases}$$

Example

$$\begin{cases} -\frac{1}{3}x_1 + x_2 + \frac{1}{3}x_4 = 1 \\ \frac{7}{3}x_1 + x_3 - \frac{1}{3}x_4 = 7 \end{cases}$$

Multiply first row by -3 and the second one by $\frac{3}{7}$.

$$\begin{cases} x_1 - 3x_2 - x_4 = -3 \\ x_1 + \frac{3}{7}x_3 - \frac{1}{7}x_4 = 3 \end{cases}$$

Now $\frac{1}{-1/3} \leqslant \frac{7}{7/3}$ but unlike the previous case, subtracting the first equation from the second one leads to an infeasible basic set $\{1, 3\}$ with $\overline{x}_{\{1,3\}} = (-3, 0, 14, 0) \not\geqslant 0$. Therefore we need to choose **the smallest ratio among the positive ones**. The only choice is $\frac{7}{7/3}$. This corresponds to the second equation, i.e. the second element from $\mathcal{B}_2 = \{2, 3\}$ leaves and $s = 1$ enter the new basic set $\mathcal{B}_3 = \{1, 2\}$.

Example

$$\begin{cases} x_1 - 3x_2 \phantom{+ \frac{3}{7}x_3} - x_4 = -3 \\ x_1 + \frac{3}{7}x_3 - \frac{1}{7}x_4 = 3 \end{cases}$$

The new basic set is $\mathcal{B}_3 = \{1, 2\}$. Subtract the second equation from the first one

$$\begin{cases} x_1 + \frac{3}{7}x_3 - \frac{1}{7}x_4 = 3 \\ x_2 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = 2 \end{cases}$$

and substitute the result to $f(x) = -x_1 - 2x_2$

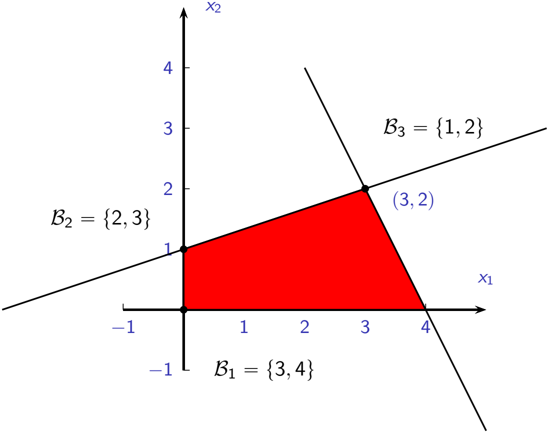
$$f(x) = -7 + \frac{5}{7}x_3 + \frac{3}{7}x_4.$$

Making x_3 or x_4 a basic variable would increase the value of f.

Example

Therefore the basic set $\mathcal{B}_3 = \{1, 2\}$ corresponds to a vertex $\overline{x}_{\{1,2\}} = (3, 2, 0, 0)$ in which function f attains minimum on the feasible region, i.e. $\overline{x}_{\{1,2\}} = (3, 2, 0, 0)$ is an optimal solution.

Example



maximize $x_1 + 2x_2$

$$\begin{cases} x_1 & \geqslant & 0 & & \\ x_2 & \geqslant & 0 & & \\ 2x_1 & + & x_2 & \leqslant & 8 \\ -x_1 & + & 3x_2 & \leqslant & 3 \end{cases}$$

Simplex Method

Given a linear programming problem in the standard form $f((x_1, \ldots, x_n)) = c_1x_1 + \ldots + c_nx_n \longrightarrow min$ under the constraints $Ax = b,\ x \geqslant 0$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

and $r(A) = m$ proceed as follows.

Simplex Method

1) build a simplex tableau $\left[\begin{array}{cccc|c} c_1 & c_2 & \cdots & c_n & 0 \\ \hline a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$ we will refer to the part above the vertical line as the upper part and to the other as the lower part,
2) find some basic feasible set $\mathcal{B} = \{i_1, \ldots, i_m\},\ i_1 < i_2 < \ldots < i_m$,
3) using elementary row operations (adding or subtracting the upper row from rows in the lower part is not allowed) bring the simplex tableau to the form

Simplex Method

$$
\begin{array}{c}
\begin{array}{cccccccccccc|c}
1 & & i_1 & & i_2 & & i_{m-1} & & i_m & & n & & \\
\end{array} \\
\left[
\begin{array}{ccccccccccc|c}
c'_1 & & 0 & & 0 & & 0 & & 0 & & c'_n & c' \\
\hline
a'_{11} & \dots & 1 & \dots & 0 & \dots\dots & 0 & \dots & 0 & \dots & a'_{1n} & b'_1 \\
a'_{21} & \dots & 0 & \dots & 1 & \dots\dots & 0 & \dots & 0 & \dots & a'_{2n} & b'_2 \\
a'_{31} & \dots & 0 & \dots & 0 & \dots\dots & 0 & \dots & 0 & \dots & a'_{3n} & b'_3 \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots & \vdots \\
a'_{(m-2)1} & \dots & 0 & \dots & 0 & \dots\dots & 0 & \dots & 0 & \dots & a'_{(m-2)n} & b'_{m-2} \\
a'_{(m-1)1} & \dots & 0 & \dots & 0 & \dots\dots & 1 & \dots & 0 & \dots & a'_{(m-1)n} & b'_{m-1} \\
a'_{m1} & \dots & 0 & \dots & 0 & \dots\dots & 0 & \dots & 1 & \dots & a'_{mn} & b'_m
\end{array}
\right]
\end{array}
$$

i.e. the submatrix of the lower part of the simplex tableau consisting of columns $i_1, \dots, i_m$ is the identity matrix and the coefficients of the objective function corresponding to the basic variables $x_{i_1}, \dots, x_{i_m}$ are zero.

Simplex Method

4) let $s \in \{1, \ldots, n\}$ be such that $c'_s = \min\{c'_1, c'_2, \ldots, c'_n\}$, i.e. let s be the number of the column with **the smallest** coefficient c'_i,
5) if $c'_s \geqslant 0$ (i.e. all c'_i are non-negative) then STOP, $-c'$ is the minimal value of the objective function and the optimal solution is $\overline{x}_{\mathcal{B}}$,
6) if the set $\{a'_{is} \mid a'_{is} > 0,\ i = 1, \ldots, m\}$ is empty, i.e. all entries in the lower part of the s-th column of the simplex tableau are non-positive then STOP, the objective function attains no minimum on the feasible region,
7) let $r \in \{1, \ldots, m\}$ be given by $\frac{b'_r}{a'_{rs}} = \min\{\frac{b'_i}{a'_{is}} \mid a'_{is} > 0, i = 1, \ldots, m\}$, i.e. let r be the number of the equation in the simplex tableau with the smallest positive ratio $\frac{b'_i}{a'_{is}}$,

Simplex Method

$$
\begin{array}{cccccccccccc|c}
1 & & i_1 & & s & & i_{m-1} & & i_m & & n & & \\
\end{array}
$$

$$
\left[\begin{array}{ccccccccccc|c}
c'_1 & & 0 & & c'_s & & 0 & & 0 & & c'_n & c' \\
\hline
a'_{11} & \dots & 1 & \dots & a'_{1s} & \dots\dots & 0 & \dots & 0 & \dots & a'_{1n} & b'_1 \\
a'_{21} & \dots & 0 & \dots & a'_{2s} & \dots\dots & 0 & \dots & 0 & \dots & a'_{2n} & b'_2 \\
a'_{31} & \dots & 0 & \dots & a'_{3s} & \dots\dots & 0 & \dots & 0 & \dots & a'_{3n} & b'_3 \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots & \vdots \\
a'_{(m-2)1} & \dots & 0 & \dots & a'_{(m-2)s} & \dots\dots & 0 & \dots & 0 & \dots & a'_{(m-2)n} & b'_{m-2} \\
a'_{(m-1)1} & \dots & 0 & \dots & a'_{(m-1)s} & \dots\dots & 1 & \dots & 0 & \dots & a'_{(m-1)n} & b'_{m-1} \\
a'_{m1} & \dots & 0 & \dots & a'_{ms} & \dots\dots & 0 & \dots & 1 & \dots & a'_{mn} & b'_m
\end{array}\right]
$$

$$
\frac{b'_r}{a'_{rs}} = \min\{\frac{b'_i}{a'_{is}} \mid a'_{is} > 0, i = 1, \dots, m\}
$$

Simplex Method

8) the r-th element of $\mathcal{B}$ (i.e. i_r) is removed and s enters the basic set $\mathcal{B}$,
9) go to step 3).

Example

Now we can redo our first example using simplex tableau.
Recall

$$-x_1 - 2x_2 \longrightarrow min$$

$$\begin{cases} 2x_1 + x_2 + x_3 = 8 \\ -x_1 + 3x_2 + x_4 = 3 \end{cases}$$

and $x_1, x_2, x_3, x_4 \geqslant 0$.
Choose basic feasible set $\mathcal{B} = \{3, 4\}$ and write the simplex tableau:

$$\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} -1 & -2 & 0 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 8 \\ -1 & 3 & 0 & 1 & 3 \end{array}\right]$$

It is already in the form from step 3) (i.e. in the upper row there are zeroes in the 3-th and 4-th column and the submatrix of the lower part consisting of columns $3, 4$ is the identity matrix).

Example

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc|c} -1 & -2 & 0 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 8 \\ -1 & 3 & 0 & 1 & 3 \end{array}\right] \end{array}$$

The smallest coefficient of the objective function is $c'_2 = -2$ and hence $s = 2$.
Compute ratios of the entries in the last column and in the second one.

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc|c} -1 & -2 & 0 & 0 & 0 \\ \hline 2 & \color{red}{1} & 1 & 0 & \color{red}{8} \\ -1 & \color{red}{3} & 0 & 1 & \color{red}{3} \end{array}\right] \end{array}$$

$$\frac{b'_2}{a'_{s2}} = \frac{3}{3} = \min\{\frac{8}{1}, \frac{3}{3}\}$$

Example

$$\frac{b'_2}{a'_{s2}} = \frac{3}{3} = \min\{\frac{8}{1}, \frac{3}{3}\}$$

The smallest ratio is provided by the second row so $r = 2$. Therefore the second element of $\mathcal{B} = \{3, 4\}$ leaves and $s = 2$ enters the basic set. For $\mathcal{B} = \{2, 3\}$ bring the simplex tableau into the form described in step 3).

$$\begin{array}{c} \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \end{array} \\ \left[\begin{array}{cccc|c} -1 & -2 & 0 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 8 \\ -1 & 3 & 0 & 1 & 3 \end{array}\right] \end{array} \xrightarrow{r_2/3} \begin{array}{c} \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \end{array} \\ \left[\begin{array}{cccc|c} -1 & -2 & 0 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 8 \\ -\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \end{array}\right] \end{array} \xrightarrow[r_1 - r_2]{r_0 + 2r_2}$$

$$\begin{array}{c} \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \end{array} \\ \left[\begin{array}{cccc|c} -\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\ \hline \frac{7}{3} & 0 & 1 & -\frac{1}{3} & 7 \\ -\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \end{array}\right] \end{array} \xrightarrow{r_1 \leftrightarrow r_2} \begin{array}{c} \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \end{array} \\ \left[\begin{array}{cccc|c} -\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\ \hline -\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \\ \frac{7}{3} & 0 & 1 & -\frac{1}{3} & 7 \end{array}\right] \end{array}$$

Example

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc|c} -\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\ \hline -\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \\ \color{red}{\frac{7}{3}} & 0 & 1 & -\frac{1}{3} & \color{red}{7} \end{array}\right] \end{array}$$

Now $c'_1 = -\frac{5}{3} < c'_4 = \frac{2}{3}$ hence $s = 1$.
In the first column only one number is positive, that is the smallest ratio is $\frac{7}{7/3}$ hence $r = 2$. The second element from $\mathcal{B} = \{2, 3\}$ leaves and $s = 1$ enters the basic set.
Now $\mathcal{B} = \{1, 2\}$.

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc|c} -\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\ \hline -\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \\ \frac{7}{3} & 0 & 1 & -\frac{1}{3} & 7 \end{array}\right] \end{array} \xrightarrow[r_1 \leftrightarrow r_2]{\frac{3}{7} r_2} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc|c} -\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\ \hline 1 & 0 & \frac{3}{7} & -\frac{1}{7} & 3 \\ -\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \end{array}\right] \end{array}$$

Example

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc|c} -\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\ \hline 1 & 0 & \frac{3}{7} & -\frac{1}{7} & 3 \\ -\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \end{array}\right] \end{array} \xrightarrow[r_2+\frac{1}{3}r_1]{r_0+\frac{5}{3}r_1} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc|c} 0 & 0 & \frac{5}{7} & \frac{3}{7} & 7 \\ \hline 1 & 0 & \frac{3}{7} & -\frac{1}{7} & 3 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & 2 \end{array}\right] \end{array}$$

Since $c_i' \geqslant 0$ for $i = 1, 2, 3, 4$ we have arrived at an optimal solution which is $\overline{x}_{\{1,2\}} = (3, 2, 0, 0)$ and the minimal value is -7.

Remarks

i) in step 2) one can guess a basic feasible set or solve an auxiliary linear programming problem to find one,

ii) row operations change neither the feasible region nor the value of the objective function on the feasible set,

iii) in step 4), choosing the smallest (negative) value of c'_s implies that we do not increase the objective function,

iv) if all elements in the lower part of the s-th column are non-positive (step 6)), we can increase arbitrarily the variable x_s staying in the feasible region while decreasing the objective function,

v) the objective function is equal to

$$f((x_1, \ldots, x_n)) = c'_1 x_1 + \ldots + c'_n x_n - c',$$

where $c'_{i_j} = 0$ for $j = 1, \ldots, m$.

Remarks

$$
\begin{array}{ccccccccccccc|c}
1 & & i_1 & & s & & i_{m-1} & & i_m & & n & & \\
\end{array}
$$

$$
\left[
\begin{array}{ccccccccccc|c}
c'_1 & & 0 & & c'_s & & 0 & & 0 & & c'_n & c' \\
\hline
a'_{11} & \dots & 1 & \dots & a'_{1s} & \dots\dots & 0 & \dots & 0 & \dots & a'_{1n} & b'_1 \\
a'_{21} & \dots & 0 & \dots & a'_{2s} & \dots\dots & 0 & \dots & 0 & \dots & a'_{2n} & b'_2 \\
a'_{31} & \dots & 0 & \dots & a'_{3s} & \dots\dots & 0 & \dots & 0 & \dots & a'_{3n} & b'_3 \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots & \vdots \\
a'_{(m-2)1} & \dots & 0 & \dots & a'_{(m-2)s} & \dots\dots & 0 & \dots & 0 & \dots & a'_{(m-2)n} & b'_{m-2} \\
a'_{(m-1)1} & \dots & 0 & \dots & a'_{(m-1)s} & \dots\dots & 1 & \dots & 0 & \dots & a'_{(m-1)n} & b'_{m-1} \\
a'_{m1} & \dots & 0 & \dots & a'_{ms} & \dots\dots & 0 & \dots & 1 & \dots & a'_{mn} & b'_m
\end{array}
\right]
$$

Move terms involving x_s to the right hand side of all equations. Set $x_i = 0$ for $i \notin \mathcal{B} \cup \{s\}$. For any positive value of x_s the system of linear equations in variables $x_{i_1}, \dots, x_{i_m}$ has a non-negative solution. That is, by increasing x_s we decrease the value of the objective function.

Example

Maximize the value $x_1 + 2x_2$ under the constraints

$$\begin{cases} x_1 & \geqslant & 0 \\ x_2 & \geqslant & 0 \\ -x_1 & + & x_2 & & \leqslant & 1 \end{cases}$$

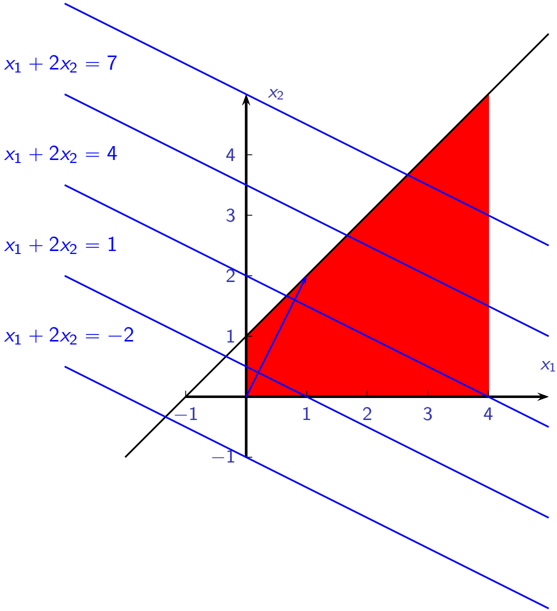
The standard form of this linear programming problem is
$f(x_1, x_2, x_3) = -x_1 - 2x_2 \longrightarrow \mathit{min}$ under the constraints

$$\begin{cases} -x_1 & + & x_2 & + & x_3 & = & 1 \end{cases}, \; x_1, x_2, x_3 \geqslant 0$$

Build the simplex tableau

$$\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc|c} -1 & -2 & 0 & 0 \\ \hline -1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

Example



maximize $x_1 + 2x_2$

$$\begin{cases} x_1 & \geqslant & 0 \\ x_2 & \geqslant & 0 \\ -x_1 + x_2 & \leqslant & 1 \end{cases}$$

no optimal solution

Example

Let $\mathcal{B} = \{3\}$ be a basic feasible set.

$$\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc|c} -1 & -2 & 0 & 0 \\ \hline -1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

Then $s = 2$ since $c'_2 = -2 < -1 = c'_1$. In the second column, in the lower part, there is only one positive element therefore $r = 1$. The new basic set is $\mathcal{B} = \{2\}$.

$$\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc|c} -1 & -2 & 0 & 0 \\ \hline -1 & 1 & 1 & 1 \end{array}\right] \end{array} \xrightarrow{r_0+2r_1} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc|c} -3 & 0 & 2 & 2 \\ \hline -1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

Then $s = 1$ and in the first column, in the lower part, all entries are non-positive. Therefore the objective function does not admit its minimum over the feasible region. In other words, there is no optimal solution.

Example

To see this, set $x_i = 0$ for $i \notin \mathcal{B} \cup \{s\} = \{1, 2\}$, i.e. $x_3 = 0$. Then

$$x_2 = 1 + x_1,$$

where the objective function is of the form

$$f((x_1, x_2, x_3)) = -3x_1 + 2x_3 - 2 = -3x_1 - 2.$$

When x_1 grows to $+\infty$ the objective function decreases to $-\infty$.

How to Find A Basic Feasible Set?

Given a linear programming problem in the standard form $f(x_1, \ldots, x_n) = c_1 x_1 + \ldots + c_n x_n \longrightarrow \mathit{min}$ under the constraints $Ax = b,\ x \geqslant 0$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

with $b \geqslant 0$ introduce auxiliary variables $y_1, \ldots, y_m$ and consider a linear programming problem in $\mathbb{R}^{n+m}$ in the standard form $g((x_1, \ldots, x_n, y_1, \ldots, y_m)) = y_1 + \ldots + y_m \longrightarrow \mathit{min}$ under the constraints $A'x' = b, x' \geqslant 0$ where

$$A' = [A|I_m] \in M(m \times (n+m); \mathbb{R}) \text{ and } x' = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_m \end{bmatrix},$$

How to Find A Basic Feasible Set?

where $I_m \in M(m \times m; \mathbb{R})$ is m-by-m identity matrix.
Solve the auxiliary problem using simplex method starting from the basic feasible set $\mathcal{B} = \{n+1, \dots, n+m\}$. It has always an optimal solution.

If the minimum of the function g is non-zero then the feasible region of the original problem is empty (there are no vertices).

Otherwise, the feasible region is non-empty. Let $\mathcal{B}$ be the basic feasible set corresponding to a optimal solution of the auxiliary problem.

How to Find A Basic Feasible Set? (continued)

There are two separate cases:

i) $\mathcal{B} \subset \{1, \ldots, n\}$, i.e. the basic feasible set $\mathcal{B}$ is also a basic feasible set of the original problem,

ii) if $n + j \in \mathcal{B}$, i.e. y_j is a basic variable, then there exists $a'_{ij} \neq 0$ for some $i \in \{1, \ldots, n\}$ (in the simplex tableaux where $c'_i = 0$ for all $i \in \mathcal{B}$) and the set $\mathcal{B}' = (\mathcal{B} \cup \{i\}) - \{n + j\}$ is also a basic feasible set of the auxiliary problem with the same basic solution as for $\mathcal{B}$.

If $a'_{ij} = 0$ for all $i \in \{1, \ldots, n\}$ then $r(A) < m$ which contradicts the assumption. Repeating step *ii*) one can make all auxiliary variables non-basic.