Linear Algebra

Lecture 7 - Application of Determinants

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Determinant and Linear Dependence

Recall that elementary operations on vectors preserve the property of being linearly independent.

Proposition

Let $A \in M(n \times n; \mathbb{R})$. The following conditions are equivalent:

i) $\det A \neq 0$,

ii) *rows of matrix A form a linearly independent set,*

iii) *columns of matrix A form a linearly independent set.*

Recall that n linearly independent vectors in $\mathbb{R}^n$ form a basis.

Example

Example

Take matrix A and use elementary row operations to get an upper-triangular matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = B$$

Then $\det A = \det B = 0$. The rows are linearly dependent

$$(1, -1, 1) - (2, 0, 3) + (1, 1, 2) = (0, 0, 0).$$

The columns are linearly dependent

$$-3(1, 2, 1) - (-1, 0, 1) + 2(1, 3, 2) = (0, 0, 0).$$

Identity Matrix

Definition

The **identity matrix** $I_n \in M(n \times n; \mathbb{R})$ is defined by

$$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}.$$

That is, it has $1'$s on the diagonal and 0's elsewhere.

Note that for any $A \in M(n \times n; \mathbb{R})$ the following holds

$$I_n A = A I_n = A,$$

that is I_n is a neutral element with respect to matrix multiplication. This follows also from the fact that $M(id_{\mathbb{R}^n})^{\mathcal{A}}_{\mathcal{A}} = I_n$ for any basis $\mathcal{A}$ of $\mathbb{R}^n$.

Invertible Matrix

Definition

A matrix $A \in M(n \times n; \mathbb{R})$ is called **invertible** if there exists matrix $B \in M(n \times n; \mathbb{R})$ such that $AB = I_n$. Such matrix B is unique and it satisfies the equality $BA = I_n$. The matrix B is called the inverse of A and is denoted A^{-1}, that is

$$AA^{-1} = A^{-1}A = I_n.$$

Examples

Example

If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Example

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$.

Proposition

Let $\mathcal{A} = (v_1, \ldots, v_n)$ and $\mathcal{B} = (w_1, \ldots, w_n)$ be ordered bases of vector space V. Let M be the change-of-coordinate matrix from the basis $\mathcal{A}$ to the basis $\mathcal{B}$, that is $M = M(\mathrm{id})^{\mathcal{B}}_{\mathcal{A}}$. Let N be the change-of-coordinate matrix from the basis $\mathcal{B}$ to the basis $\mathcal{A}$, that is $N = M(\mathrm{id})^{\mathcal{A}}_{\mathcal{B}}$. Then $N = M^{-1}$.

Proof.

It is enough to use the formula relating composition of linear transformations with matrix multiplication and the uniqueness of the inverse.

$$MN = M(\mathrm{id})^{\mathcal{B}}_{\mathcal{A}} M(\mathrm{id})^{\mathcal{A}}_{\mathcal{B}} = M(\mathrm{id})^{\mathcal{B}}_{\mathcal{B}} = I_n.$$

□

Example

Let $V = \mathbb{R}^2$, $\mathcal{A} = ((2,1),(5,3)), \mathcal{B} = st = ((1,0),(0,1))$. Then $M = M(\mathrm{id})^{st}_{\mathcal{A}} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $N = M(\mathrm{id})^{\mathcal{A}}_{st} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Example (continued)

Let $V = \mathbb{R}^2$, $\mathcal{A} = ((2,1),(5,3)), \mathcal{B} = st = ((1,0),(0,1))$. Then $M = M(\mathrm{id})^{st}_{\mathcal{A}} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $N = M(\mathrm{id})^{\mathcal{A}}_{st} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.
For example, take vector $v = (3,1)$. It's coordinates relative to the standard basis are $3, 1$ that is $(3,1) = 3(1,0) + 1(0,1)$. To compute coordinates of v relative to the basis $\mathcal{A}$ we use the change-of-coordinate matrix $N = M(\mathrm{id})^{\mathcal{A}}_{st}$.

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

The coordinates of v relative to the basis $\mathcal{A}$ are $4, -1$ that is

$$(3,1) = 4(2,1) - 1(5,3).$$

Determinants and Invertible Matrices

Theorem

Let $A \in M(n \times n; \mathbb{R})$. Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and let $\mathcal{A}, \mathcal{B}$ be bases of $\mathbb{R}^n$ such that $M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = A$. The following conditions are equivalent:

i) *the matrix A is invertible,*

ii) $\det A \neq 0$,

iii) *rows of A form a linearly independent set,*

iv) *columns of A form a linearly independent set,*

v) *for any* $K = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$ *if* $AK = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ *then* $K = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$,

vi) *the linear transformation φ is injective,*

vii) *the linear transformation φ is surjective,*

viii) *the linear transformation φ is bijective (invertible).*

Computing the Inverse

For any $A = [a_{ij}], B = [b_{ij}] \in M(n \times n; \mathbb{R})$ denote by $[A|B]$ the matrix

$$\left[\begin{array}{ccc|ccc} a_{11} & \dots & a_{1n} & b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} & b_{n1} & \dots & b_{nn} \end{array}\right] \in M(n \times 2n; \mathbb{R}).$$

Theorem

Matrix A is invertible if and only if matrix $[A|I_n]$ can be transformed by elementary row operations to the matrix $[I_n|B]$. Then $B = A^{-1}$.

Proof.

Use multiplication by elementary matrices (cf. Lecture 5). □

Example

Let $A = \left[\begin{array}{ccc} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$. Then $\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_1 - r_2}$
$\left[\begin{array}{ccc|rrr} 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|rrr} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_3 - r_2}$
$\left[\begin{array}{ccc|rrr} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|rrr} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & 2 & 0 \end{array}\right].$

Therefore

$$A^{-1} = \left[\begin{array}{rrr} 1 & -1 & 0 \\ 1 & -2 & 1 \\ -1 & 2 & 0 \end{array}\right].$$

Rank of Matrix

Recall

Definition

Let $A \in M(m \times n; \mathbb{R})$. The rank of A is the dimension of the space $\mathrm{lin}(w_1, \ldots, w_m)$ where $w_1, \ldots, w_m \in \mathbb{R}^n$ are rows of A. The rank of A is denoted $r(A)$.

Example

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 3 & 4 \\ 1 & 3 & 1 & 2 \end{bmatrix} \xrightarrow[r_3 - r_1]{r_2 - 3r_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The last matrix is in an echelon form with two non-zero rows therefore $r(A) = \dim \mathrm{lin}((1,2,1,1),(3,7,3,4),(1,3,1,2)) =$
$= \dim \mathrm{lin}((1,2,1,1),(0,1,0,1)) = 2.$

Rank of Matrix

Theorem

For any matrix $A \in M(m \times n; \mathbb{R})$ the following numbers are equal:

i) $\dim \operatorname{lin}(w_1, \ldots, w_m)$ *where* $w_1, \ldots, w_m$ *are rows of* A,

ii) $\dim \operatorname{lin}(k_1, \ldots, k_n)$ *where* $k_1, \ldots, k_n$ *are columns of* A,

iii) *the number of columns (or rows) of a maximal square submatrix* B *of* A *such that* $\det B \neq 0$.

Proof

Matrix A can be put into a reduced echelon form by elementary row operations, and then, by elementary operations on columns, it can be put into the form

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Elementary row and column operations do not change those three numbers. Therefore the rank of A is equal to the number of pivots in an echelon form.

Example

Example

Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 3 & 4 \\ 1 & 3 & 1 & 2 \end{bmatrix}$. It can be checked that

$\det \begin{bmatrix} 2 & 1 & 1 \\ 7 & 3 & 4 \\ 3 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 4 \\ 1 & 3 & 2 \end{bmatrix} =$
$\det \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 1 & 3 & 1 \end{bmatrix} = 0$. But $\det \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = 1 \neq 0$, hence $r(A) = 2$.

Kronecker-Capelli Theorem

Consider a system of linear equations and two associated matrices

$$U:\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}\right.$$

$$A=\left[\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array}\right],\ B=\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Kronecker-Capelli Theorem (continued)

Theorem (Kronecker-Capelli)

i) *the system U has a solution if and only if $r(A) = r(B)$,*

ii) *if the system U has a solution then exactly $n - r(A)$ variables are free variables,*

iii) *if $(s_1, \ldots, s_n) \in \mathbb{R}^n$ is any solution of U and W is the subspace of all solutions of a homogeneous system of linear equations given by the matrix A then solutions of U are of the form $(s_1, \ldots, s_n) + W = \{(s_1, \ldots, s_n) + w \mid w \in W\}$.*

Proof.

Adding one column to a matrix can only increase its rank by at most 1. If $r(B) = r(A) + 1$ then in the echelon form of B there is a pivot in the column of constant terms. The pivots correspond to dependent variables and the number of pivots is equal to the rank of the matrix. The difference of any two solutions of U is a solution of the homogeneous system of linear equations associated to the matrix A. $\square$

Matrix Inverse Formula

Let $A \in M(n \times n; \mathbb{R})$. The **adjugate** matrix of the matrix A is given by **(note the transposition!)**

$$\operatorname{adj}(A) = \begin{bmatrix} (-1)^{1+1}\det A_{11} & (-1)^{1+2}\det A_{12} & \cdots & (-1)^{1+n}\det A_{1n} \\ (-1)^{2+1}\det A_{21} & (-1)^{2+2}\det A_{22} & \cdots & (-1)^{2+n}\det A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}\det A_{n1} & (-1)^{n+2}\det A_{n2} & \cdots & (-1)^{n+n}\det A_{nn} \end{bmatrix}^{\mathsf{T}}.$$

Theorem

Let $A \in M(n \times n; \mathbb{R})$ be an invertible matrix. Then

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}(A).$$

Proof.

The equality $A\frac{1}{\det A}\operatorname{adj}(A) = I_n$ can be checked directly using the Laplace expansion. $\square$

Example

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\text{adj}(A) = \begin{bmatrix} (-1)^{1+1}\det A_{11} & (-1)^{1+2}\det A_{12} \\ (-1)^{2+1}\det A_{21} & (-1)^{2+2}\det A_{22} \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^{\mathsf{T}} =$$

$$= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Hence

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For example $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Cramer's Rule

Let U be a system of linear equations with n unknowns and n equations:

$$U : \begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = b_n \end{cases}$$

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ be the associated matrix of coefficients and let $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ be the matrix of constant terms.

The system U can be written as $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$.

Cramer's Rule (continued)

Therefore, if $\det A \neq 0$ the system U has exactly one solution given by $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$.

Theorem (Cramer's Rule)

If $\det A \neq 0$ then the unique solution of the system U is given by $x_i = \frac{\det A_i}{\det A}$ for $i = 1, \ldots, n$, where A_i is the matrix A with i-th column replaced by B.

Proof.

Use the Laplace expansion and the inverse matrix formula. $\square$

Example

Let

$$U : \begin{cases} 2x_1 & + & 3x_2 & = -1 \\ 3x_1 & + & 4x_2 & = -3 \end{cases}$$

Then

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}, \; A_1 = \begin{bmatrix} -1 & 3 \\ -3 & 4 \end{bmatrix}, \; A_2 = \begin{bmatrix} 2 & -1 \\ 3 & -3 \end{bmatrix}.$$

Therefore, $x_1 = \frac{\det A_1}{\det A} = \frac{5}{-1} = -5, \; x_2 = \frac{\det A_2}{\det A} = \frac{-3}{-1} = 3.$

Matrix Algebra

Remarks

i) *if* $A, B \in M(n \times n; \mathbb{R})$ *and* $\det A \neq 0, \det B \neq 0$ *then the matrix* AB *is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$,

ii) $(AB)^{\intercal} = B^{\intercal}A^{\intercal}$,

iii) *if* $A \in M(n \times n; \mathbb{R})$ *and* $\det A \neq 0$ *then the matrix* $A^{\intercal}$ *is invertible and* $(A^{\intercal})^{-1} = (A^{-1})^{\intercal}$,

iv) *for* $n > 0$ *define*

$$A^n = A \cdots A \ (n - times),$$

if $\det A \neq 0$ *for* $n < 0$ *define*

$$A^n = (A^{-1})^{-n}$$

and $A^0 = I$.

Matrix Algebra (continued)

Remarks

iv) *The following*

$$A^n A^m = A^{n+m},$$
$$(A^n)^m = A^{nm},$$

hold for any integers m, n,

v) *note that unless* $AB = BA$*, in general,* $(AB)^n = (AB)(AB)\cdots(AB) \neq A^n B^n$.