Linear Algebra

Lecture 3 - Linear Independence and Bases

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Linearly (In)dependent Vectors

Let V be a vector space.

Definition

Vectors $v_1, \ldots, v_k \in V$ are said to be **linearly dependent** if there exist real numbers $\alpha_1, \ldots, \alpha_k$, not all of which are 0 such that

$$\alpha_1 v_1 + \ldots + \alpha_k v_k = \mathbf{0}.$$

Vectors $v_1, \ldots, v_k \in V$ are said to be **linearly independent** if they are not linearly dependent.

By definition, vectors $v_1, \ldots, v_k$ are linearly independent if $\alpha_1 v_1 + \ldots + \alpha_k v_k = \mathbf{0}$ implies that $\alpha_1 = \ldots = \alpha_k = 0$.

Linear independence does not depend on the order of vectors hence we may talk about independent (finite) sets. We assume that empty set is linearly independent.

Examples

i) vectors $(1,1,2),(1,1,0),(2,2,1) \in \mathbb{R}^3$ are linearly dependent because $(1,1,2)+3(1,1,0)-2(2,2,1)=(0,0,0)$,

ii) vectors $(1,2),(0,1) \in \mathbb{R}^2$ are linearly independent,

iii) vectors $(1,2),(0,1),(0,0) \in \mathbb{R}^2$ are linearly dependent,

iv) vectors $(1,2),(2,4) \in \mathbb{R}^2$ are linearly dependent,

v) vector $\varepsilon_i = (0,\ldots,0,\overset{\overset{i}{\vee}}{1},0,\ldots,0) \in \mathbb{R}^n$ with 1 at the i-th coordinate and 0 elsewhere is called **unit vector**. Vectors $\varepsilon_1,\ldots,\varepsilon_n$ are linearly independent.

Properties

Proposition

Single vector $v \in V$ is linearly independent if and only if $v \neq \mathbf{0}$.

Proposition

Any subset of linearly independent vectors is linearly independent.

Proposition

A set of at least two vectors is linearly dependent if and only if one vector is a linear combination of the others.

Steinitz's Theorem

Theorem (Steinitz's Theorem)

If vectors $w_1, \ldots, w_m \in \operatorname{lin}(v_1, \ldots, v_n)$ *are linearly independent then* $m \leqslant n$*.*

We postpone the proof of this theorem until the end of the lecture.

For example, since $\mathbb{R}^n = \operatorname{lin}(\varepsilon_1, \ldots, \varepsilon_n)$ any independent set of vectors in $\mathbb{R}^n$ has at most n elements.

Basis

Definition

Vectors $v_1, \ldots, v_n \in V$ form a **basis** of the vector space V if:

i) they are linearly independent,

ii) $V = \text{lin}(v_1, \ldots, v_n)$, i.e. they span V.

In general, a vector space can have many bases.

Examples

i) vectors $(1,2),(0,1) \in \mathbb{R}^2$ form a basis of $\mathbb{R}^2$,

ii) vectors $(1,0),(0,1) \in \mathbb{R}^2$ form a basis of $\mathbb{R}^2$,

iii) vectors $\varepsilon_1, \ldots, \varepsilon_n$ form a basis of $\mathbb{R}^n$. It is called the **standard basis**,

iv) the set of solutions of a homogeneous system of linear equations is a vector space, its basis can be computed by substituting subsequently each free variable with 1 and the other free variables with $0'$s.

Example

Consider the following general solution of a homogeneous system of linear equations: $\begin{cases} x_1 &= 2x_2 + 4x_4 + x_5 \\ x_3 &= \quad\;\;\; - 3x_4 - x_5 \end{cases}$

The free variables are x_2, x_4 and x_5. By substituting $x_2 = 1, x_4 = x_5 = 0$ and then $x_4 = 1, x_2 = x_5 = 0$ and $x_5 = 1, x_2 = x_4 = 0$ we get three vectors $(2, 1, 0, 0, 0), (4, 0, -3, 1, 0), (1, 0, -1, 0, 1)$ which form a basis of the space of all solution. In fact, every solution can be uniquely written in the form

$$(2x_2 + 4x_4 + x_5, x_2, -3x_4 - x_5, x_4, x_5) =$$
$$x_2(2, 1, 0, 0, 0) + x_4(4, 0, -3, 1, 0) + x_5(1, 0, -1, 0, 1), \;\; x_2, x_4, x_5 \in \mathbb{R}.$$

Dimension

Proposition

If a vector space V has a basis consisting of n vectors than any other basis has n vectors.

Definition

A vector space V is said to be n-dimensional if it has basis consisting of n vectors. We write $\dim V = n$ and say n is dimension of V. It is assumed that $\dim\{\mathbf{0}\} = 0$. A finite-dimensional vector space is a space of dimension $0, 1, 2 \ldots$, otherwise it is infinite-dimensional and write $\dim V = \infty$.

Examples

i) $\dim \mathbb{R}^n = n$,

ii) if $V_U \subset \mathbb{R}^n$ is a subspace consisting of solutions of a homogeneous system of linear equations U then $\dim V_U =$the number of free variables,

iii) $\dim \mathbb{R}^\infty = \infty$ since it contains arbitrarily many independent vectors.

Linear Independence and Linear Span

Proposition

Let $v_1, \dots, v_{k+1} \in V$ *and let* $v_1, \dots, v_k$ *be linearly independent vectors. Then*

$$v_1, \dots, v_{k+1} \text{ are linearly independent} \Leftrightarrow v_{k+1} \notin \operatorname{lin}(v_1, \dots, v_k).$$

Proof.

$(\Leftarrow)$ Assume that $\alpha_1 v_1 + \ldots + \alpha_{k+1} v_{k+1} = \mathbf{0}$. Then $\alpha_{k+1} = 0$, by assumption. Vectors $v_1, \dots, v_k$ are linearly independent hence $\alpha_1 = \ldots = \alpha_k = 0$. $\square$

Properties

Proposition

Let V be a vector space. The following conditions are equivalent:

i) *vectors $v_1, \ldots, v_n$ form a basis of V,*

ii) *vectors $v_1, \ldots, v_n$ form a minimal set spanning V,*

iii) *vectors $v_1, \ldots, v_n$ form a maximal linearly independent set in V.*

$i) \Rightarrow ii)$ basis is a set spanning V, if removing say v_n, makes it a smaller set spanning V, then by the previous Proposition $v_n \notin \mathrm{lin}(v_1, \ldots, v_{n-1})$,
$ii) \Rightarrow iii)$ a minimal set spanning V must be linearly independent since otherwise you could make it smaller by removing dependent vectors, it is maximal linearly independent set in V again by the previous Proposition,
$iii) \Rightarrow i)$ it is enough to show that $v_1, \ldots, v_n$ span V, if they do not, by the previous Proposition, you could make it bigger contradicting maximality.

Properties (continued)

Proposition

Let $v_1, \ldots, v_k \in V$ *are independent vectors. Then*

i) $k \leqslant \dim V$,

ii) $v_1, \ldots, v_k$ *form a basis of* V *if and only if* $k = \dim V$.

iii) *if* $W \subset V$ *is a subspace then* $\dim W \leqslant \dim V$. *If* $\dim W = \dim V$ *then* $V = W$,

iv) $\dim \operatorname{lin}(v_1, \ldots, v_k) = k$ *if and only if* $v_1, \ldots, v_k$ *is a basis of* $\operatorname{lin}(v_1, \ldots, v_k)$.

Proof.

i) by the Steinitz's Theorem,

ii) $(\Leftarrow)$ if $k = \dim V$ and $v_{k+1} \in V \setminus \operatorname{lin}(v_1, \ldots, v_k)$ then one can find $\dim V + 1$ linearly independent vectors in V,

iii) as in *ii*),

iv) this is *i*) with $V = \operatorname{lin}(v_1, \ldots, v_k)$.

$\square$

Coordinates

Proposition

Vectors $v_1, \ldots, v_n$ form a basis of V if and only if any vector $v \in V$ can be uniquely written (up to the order of summands) as $v = \alpha_1 v_1 + \ldots \alpha_n v_n$.

Proof.

($\Rightarrow$) basis spans the vector space V, hence any vector $v \in V$ is a linear combination of $v_1, \ldots, v_n$. If $v = \alpha_1 v_1 + \ldots \alpha_n v_n$ and $v = \beta_1 v_1 + \ldots \beta_n v_n$ then $\mathbf{0} = (\alpha_1 - \beta_1)v_1 + \ldots (\alpha_n - \beta_n)v_n$. This gives $\alpha_i = \beta_i$ for $i = 1, \ldots, n$.
($\Leftarrow$) By assumption $v_1, \ldots, v_n$ span the vector space V. To prove they are linearly independent take $v = \mathbf{0}$. $\square$

Coordinates (continued)

Definition

Let $\mathcal{B} = (v_1, \ldots, v_n)$ be an ordered basis of V. If $v = \alpha_1 v_1 + \ldots \alpha_n v_n$ the unique numbers $\alpha_1, \ldots, \alpha_n$ are called the **coordinates** of v relative to the basis $\mathcal{B}$.

For example, let $\mathcal{B} = (\varepsilon_1, \varepsilon_2, \varepsilon_3), \mathcal{B}' = (\varepsilon_2, \varepsilon_3, \varepsilon_1)$ and $\mathcal{B}'' = ((0,0,3),(0,2,0),(1,0,0))$ be three bases of $\mathbb{R}^3$. The coordinates of the vector $v = (1,2,3)$ relative to the basis $\mathcal{B}$ are $1,2,3$, relative to the basis $\mathcal{B}'$ are $2,3,1$ and relative to the basis $\mathcal{B}''$ are $1,1,1$ since

$$(1,2,3) = 1(1,0,0) + 2(0,1,0) + 3(0,0,1),$$

$$(1,2,3) = 2(0,1,0) + 3(0,0,1) + 1(1,0,0),$$

$$(1,2,3) = 1(0,0,3) + 1(0,2,0) + 1(1,0,0).$$

Linear Independence and Elementary Operations

Let V be a vector space.

Proposition

Assume that vectors $v_1, v_2, \ldots, v_k \in V$ are linearly independent and $\alpha \in \mathbb{R} - \{0\}$. Then

i) *the vectors $v_1 + v_2, v_2, v_3, \ldots, v_k$ are linearly independent,*

ii) *the vectors $\alpha v_1, v_2, v_3, \ldots, v_k$ are linearly independent.*

Proof.

Assume that $v_1, \ldots, v_k$ are linearly independent. The expression $\alpha_1(v_1 + v_2) + \alpha_2 v_2 + \alpha_3 v_3 + \ldots + \alpha_k v_k = \mathbf{0}$ can be rewritten as $\alpha_1 v_1 + (\alpha_1 + \alpha_2) v_2 + \alpha_3 v_3 + \ldots + \alpha_k v_k = \mathbf{0}$. By assumption $\alpha_1 = \alpha_1 + \alpha_2 = \alpha_3 = \ldots = \alpha_k = 0$ so $\alpha_i = 0$. The second case can be proven in a similar way. $\square$

Linear Independence and Elementary Operations (continued)

Corollary

Let $v_1, \ldots, v_n \in V$ and $\alpha \in \mathbb{R}$. The vectors $v_1, \ldots, v_n$ form a basis of V if and only if the vectors $v_1 + \alpha v_2, v_2, v_3, \ldots, v_n$ form a basis of V.

Proof of Steinitz's Theorem

Theorem (Steinitz's Theorem)

If vectors $w_1, \ldots, w_m \in \text{lin}(v_1, \ldots, v_n)$ *are linearly independent then* $m \leqslant n$.

Proof.

Assume that $w_1, \ldots, w_m$ are linearly independent and $m > n$. Let $a_{ij} \in \mathbb{R}$ be the numbers given by conditions

$$w_i = a_{i1}v_1 + a_{i2}v_2 + \ldots + a_{in}v_n \text{ for } i = 1, \ldots, m.$$

Let $A = [a_{ij}]$ for $i = 1, \ldots, m, j = 1, \ldots, n$ be an m-by-n matrix. Elementary row operations on A correspond to elementary operations on vectors $w_1, \ldots, w_m$. Since the matrix A has more rows than columns it has a zero row in its reduced echelon form which contradicts the assumption. $\square$