Linear Algebra

Lecture 12 - Linear Programming

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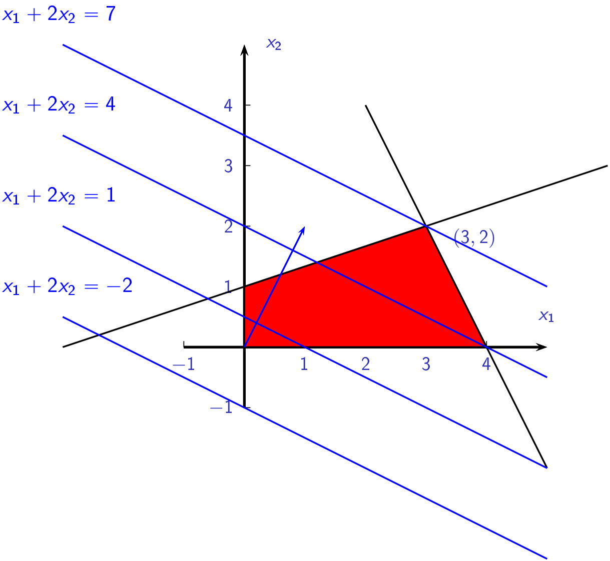
What is Linear Programming?

Example

Maximize the value $x_1 + 2x_2$ under the constraints

$$\begin{cases} x_1 & \geqslant & 0 & & \\ x_2 & \geqslant & 0 & & \\ 2x_1 & + & x_2 & \leqslant & 8 \\ -x_1 & + & 3x_2 & \leqslant & 3 \end{cases}$$

What is Linear Programming?



maximize $x_1 + 2x_2$

$$\begin{cases} x_1 & \geqslant & 0 \\ x_2 & \geqslant & 0 \\ 2x_1 + x_2 & \leqslant & 8 \\ -x_1 + 3x_2 & \leqslant & 3 \end{cases}$$

optimal solution is $(3, 2)$

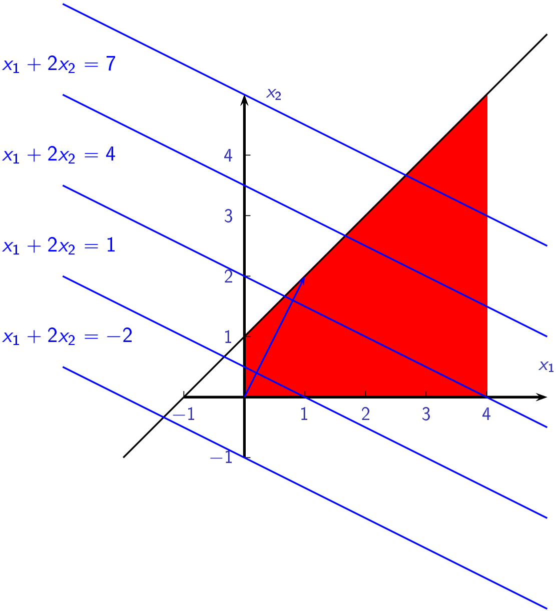
What is Linear Programming?

Example

Maximize the value $x_1 + 2x_2$ under the constraints

$$\begin{cases} x_1 & \geqslant & 0 & & \\ x_2 & \geqslant & 0 & & \\ -x_1 & + & x_2 & \leqslant & 1 \end{cases}$$

What is Linear Programming?



maximize $x_1 + 2x_2$

$$\begin{cases} x_1 & \geqslant 0 \\ x_2 & \geqslant 0 \\ -x_1 + x_2 & \leqslant 1 \end{cases}$$

no optimal solution

Economy and Economical

The second meaning of ‘economy’ in the Oxford British and World English Dictionary

Careful management of available resources.

The first meaning of ‘economical’ in the Oxford British and World English Dictionary

Giving good value or return in relation to the money, time, or effort expended.

from Greek

oikonomia=household management, housekeeping

Linear Programming Problem

Definition

Linear programming problem is a task of maximizing or minimizing a linear function (called an **objective function**) over a set $X \subset \mathbb{R}^n$ described by a finite number of linear equalities and inequalities.

That is, we look for a maximal or minimal value of the function $f((x_1, x_2, \ldots, x_n)) = c_1 x_1 + c_2 x + \ldots + c_n x_n$ on the set $X \subset \mathbb{R}^n$ of points satisfying the following conditions

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = b_2 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = b_m \end{cases}$$

i.e. m equalities and

$$\begin{cases} a'_{11}x_1 & + & a'_{12}x_2 & + & \dots & + & a'_{1n}x_n & \leqslant b'_1 \\ a'_{21}x_1 & + & a'_{22}x_2 & + & \dots & + & a'_{2n}x_n & \leqslant b'_2 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ a'_{k1}x_1 & + & a'_{k2}x_2 & + & \dots & + & a'_{kn}x_n & \leqslant b'_k \end{cases}$$

i.e. k inequalities.

Those conditions (also called **constraints**) can be written in concise form. Set

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad A' = \begin{bmatrix} a'_{11} & \cdots & a'_{1n} \\ \vdots & \ddots & \vdots \\ a'_{k1} & \cdots & a'_{kn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \; c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, b' = \begin{bmatrix} b'_1 \\ \vdots \\ b'_k \end{bmatrix}$$

The linear programming problem can be written in the form: maximize (or minimize) the function $f(x) = c^{\mathsf{T}}x$ over the set $X \subset \mathbb{R}^n$ given by

$$Ax = b, \quad A'x \leqslant b'$$

Equivalently, one can write $f(x) \longrightarrow max(resp.\ min)$ or $\max\{f(x) \mid x \in X\}$ (resp. $\min\{f(x) \mid x \in X\}$).

An inequality of type $a_1x_1 + \ldots + a_nx_n \geqslant b$ is equivalent to the inequality $-a_1x_1 - \ldots - a_nx_n \leqslant -b$.

Real Life Applications - Transportation Problem

A firm produces some goods at l supply centers and ships those goods to k markets. The cost of transporting a unit of those goods from the i-th supply center to the j-th market is a_{ij}. Each market demands at least of b_j units of those goods. Each supply center produces at most w_i units of goods.
Introduce $l \times k$ variables x_{ij} for $i = 1, \ldots, l$ and $j = 1, \ldots, k$ denoting the amount of the transport from the i-th supply center to the j-th market. We want to minimize the cost of transport and to satisfy demands of all markets. We minimize the linear function $\sum_{i=1}^{l} \sum_{j=1}^{k} k_{ij} x_{ij}$ under the constraints

$$\begin{cases} x_{11} + x_{12} + x_{13} + \ldots + x_{1k} \leqslant w_1 \\ x_{21} + x_{22} + x_{23} + \ldots + x_{2k} \leqslant w_2 \\ \vdots \\ x_{l1} + x_{l2} + x_{l3} + \ldots + x_{lk} \leqslant w_l \end{cases}$$

i.e. no supply center cannot supply more than w_i of goods and

Real Life Applications - Transportation Problem

$$\begin{cases} x_{11} + x_{21} + x_{31} + \ldots + x_{l1} \geqslant b_1 \\ x_{12} + x_{22} + x_{32} + \ldots + x_{l2} \geqslant b_2 \\ \vdots \\ x_{1k} + x_{2k} + x_{3k} + \ldots + x_{lk} \geqslant b_k \end{cases}$$

i.e. the demand of each market is satisfied. We want to transport from a supply center to a market so we assume

$$x_{ij} \geqslant 0 \text{ for } i = 1, \ldots, l \text{ and } j = 1, \ldots, k.$$

Real Life Application - Diet Problem

Suppose there are n foods available. The cost of serving per j-th food is q_j. Assume there are k nutrients and each serving of j-th type of food contains z_{ij} units of the i-th nutrient. We want to find a healthy diet minimizing its cost. Let N_i denotes the minimal amount of units of the i-th nutrient in a healthy diet. Introduce n variables $x_1, \ldots, x_n$, where x_j stands for the amount of servings of the j-th food. We minimize the function $q_1x_1 + q_2x_2 + \ldots + q_nx_n$ under the constraints

$$\begin{cases} z_{11}x_1 + z_{12}x_2 + z_{13}x_3 + \ldots + x_{1n}x_n \geqslant N_1 \\ z_{21}x_1 + z_{22}x_2 + z_{23}x_3 + \ldots + x_{2n}x_n \geqslant N_2 \\ \vdots \\ z_{k1}x_1 + z_{k2}x_2 + z_{k3}x_3 + \ldots + x_{kn}x_n \geqslant N_k \end{cases}$$

Real Life Applications - Diet Problem

If needed one may add another constraints for the minimal or maximal amount of servings of each type of food. A similar problem was considered in 1930s and 1940s in order to find an optimal diet for the US soldiers.

Real Life Applications

And many more: portfolio optimization, network design, vehicle routing.

Convex Polytopes

Definition

For any $x, y \in \mathbb{R}^n$ the **segment** joining x and y is the set $\{tx + (1-t)y \mid t \in [0,1]\}$. A set $X \subset \mathbb{R}^n$ is said to be **convex** if for any $x, y \in X$ the segment joining x and y is contained in X.

Proposition

Intersection of a finite number of convex sets is a convex set.

Definition

A **half-space** is a subset of $\mathbb{R}^n$ given by the inequality $a_1x_1 + \ldots + a_nx_n \leqslant b$. A **polytope** is a subset of $\mathbb{R}^n$ equal to a intersection of a finite number of half-spaces.

A half-space is a convex set. Therefore a polytope is a convex set.

Convex Polytopes

Suppose we are given a linear programming problem with constraints $Ax = b,\ A'x \leqslant b'$ with $f(x) = c^{\mathsf{T}}x \longrightarrow min$.

Definition

A **feasible region** (also a **feasible set)** is the set of all points $X \subset \mathbb{R}^n$ satisfying the conditions $Ax = b,\ A'x \leqslant b'$. An optimal solution is any point $\overline{x} \in X$ such that $f(\overline{x}) \leqslant f(x)$ for any $x \in X$.

A feasible region is a convex polytope. If it is bounded (i.e. contained in a ball) then there exists an optimal solution. An optimal solution may be not unique.

Supporting Hyperplane

Definition

Let $X \subset \mathbb{R}^n$ be a convex set. A **supporting hyperplane** of X is a hyperplane H given by the equation $a_1x_1 + \ldots + a_nx_n = b$ such that $H \cap X \neq \varnothing$ and $X \subset H_+$ or $X \subset H_-$ where

$$\mathrm{H}_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \ldots + a_nx_n \geqslant b\}$$

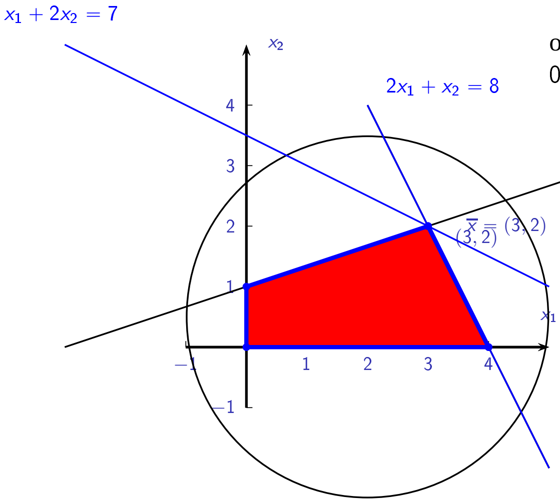
$$\mathrm{H}_- = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \ldots + a_nx_n \leqslant b\}$$

Definition

A **face** of a polytope X is the intersection of X with its **supporting hyperplane**. A face of X which is a point is called a **vertex** of X.

A face of a polytope is a polytope. Equivalently, a vertex of X can be defined as a point of X which for any $x, y \in X$ is not an interior point of the segment joining x and y (i.e. point of the segment different from x and y) - so called extremal point.

Example



optimal solution may not be unique

0$-$dimensional faces

maximize $2x_1 + x_2$

$$\begin{cases} x_1 & \geqslant & 0 \\ x_2 & \geqslant & 0 \\ 2x_1 + x_2 & \leqslant & 8 \\ -x_1 + 3x_2 & \leqslant & 3 \end{cases}$$

Optimal Solution

Theorem

An optimal solution of a linear programming problem, if it exists, belongs to a face of the feasible region.

That is, if an optimal solution exists it can be chosen to be a vertex of the feasible region.

Definition

A linear programming problem in $\mathbb{R}^n$ is in the **standard form** if the only constraints are of the type

$$Ax = b, \; x \geqslant 0,$$

and we look for the **minimum** of the objective function $f(x) = c^\intercal x$.

Standard Form

Theorem

Any linear programming problem can be brought to a standard form.

The following operations on the a linear programming data give an equivalent problem:

i) the condition $f(x) \longrightarrow max$ can be replaced by

$$-f(x) \longrightarrow min,$$

ii) the inequality $a_1x_1 + \ldots + a_nx_n \leqslant b$ can replaced by $a_1x_1 + \ldots + a_nx_n + x_{n+1} = b$ and $x_{n+1} \geqslant 0$, the inequality $a_1x_1 + \ldots + a_nx_n \geqslant b$ can replaced by $a_1x_1 + \ldots + a_nx_n - x_{n+1} = b$ and $x_{n+1} \geqslant 0$, the newly introduced variable x_{n+1} is called **slack variable**,

iii) the condition $x_i \leqslant 0$ can be replaced by $x_i' \geqslant 0$ and $x_i' = -x_i$,

iv) if there are no constraints on the variable x_i, one can introduce two slack variables $x_i^-, x_i^+ \geqslant 0$ and set $x_i = x_i^+ - x_i^-$.

Example

Bring to a standard form the following linear programming problem:
$x_1 + 2x_2 \longrightarrow max$

$$\begin{cases} x_1 \geqslant 0 \\ x_2 \geqslant 0 \\ 2x_1 + x_2 \leqslant 8 \\ -x_1 + 3x_2 \leqslant 3 \end{cases}$$

A standard form: $-x_1 - 2x_2 \longrightarrow min$

$$\begin{cases} 2x_1 + x_2 + x_3 = 8 \\ -x_1 + 3x_2 + x_4 = 3 \end{cases}$$

and $x_1, x_2, x_3, x_4 \geqslant 0$.

Example (continued)

Equivalently, it can be written $c^\mathsf{T} x \longrightarrow min,\ \ Ax = b,\ \ x \geqslant 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} 8 \\ 3 \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},\ c = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

The optimal solution is

$$\overline{x} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} \text{ and } c^\mathsf{T}\overline{x} = -7$$

Basic Set, Basic Variables

From now on we deal only with a linear programming problem in the standard form $c^T x \longrightarrow min, Ax = b, x \geqslant 0$ where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \ b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

We can assume that $r(A) = r([A|b]) = m$ (i.e. the system $Ax = b$ has solutions and no equation is redundant).

Definition

A basic set $\mathcal{B} = \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ is a set of m elements such that columns $k_{i_1}, \ldots, k_{i_m}$ of the matrix A are linearly independent. The variables $x_{i_1}, \ldots, x_{i_m}$ are called **basic variables**. The other variables are called **non-basic**.

Example

Consider a linear programming problem
$c^{\mathsf{T}}x \longrightarrow min,\ Ax = b,\ x \geqslant 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

There are $\binom{4}{2} = 6$ basic sets, i.e. every set of 2 elements is basic.

Example

Consider a linear programming problem
$c^{\mathsf{T}}x \longrightarrow min,\ \ Ax = b,\ \ x \geqslant 0$ where

$$A = \begin{bmatrix} 2 & -6 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix},\ \ b = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

There are 5 basic sets
$\mathcal{B}_1 = \{1,3\}, \mathcal{B}_2 = \{1,4\}, \mathcal{B}_3 = \{2,3\}, \mathcal{B}_4 = \{2,4\}, \mathcal{B}_5 = \{3,4\}.$

Basic Solution and Basic Feasible Solution

Definition

Let $\mathcal{B}$ be a basic set. The unique solution $\overline{x}_{\mathcal{B}} \in \mathbb{R}^n$ of the system of linear equations $Ax = b$ with $x_i = 0$ for $i \notin \mathcal{B}$ is called a **basic solution**. The basic set $\mathcal{B}$ such that $\overline{x}_{\mathcal{B}} \geqslant 0$ is called a **feasible basic set** and the solution $\overline{x}_{\mathcal{B}}$ is called a **feasible basic solution**. Otherwise the basic set $\mathcal{B}$ and the basic solution $\overline{x}_{\mathcal{B}}$ are called **infeasible**.

Theorem

Basic feasible solutions correspond to vertices of the polytope X given by the conditions $Ax = b,\ x \geqslant 0$.

Proof.

The equation $\sum_{j \in \{1,\dots,n\} - \mathcal{B}} x_j = 0$ defines a supporting hyperplane which intersects with the feasible region in exactly one point, that is in a vertex. $\square$

Example

Consider a linear programming problem
$c^{\mathsf{T}}x \longrightarrow min, \; Ax = b, \; x \geqslant 0$ where

$$A = \left[\begin{array}{rrrr} 2 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array} \right], \; b = \left[\begin{array}{c} 8 \\ 3 \end{array} \right]$$

The set $\mathcal{B} = \{3, 4\}$ is basic. The corresponding basic solution $\overline{x}_{\mathcal{B}} = \left[\begin{array}{cccc} 0 & 0 & 8 & 3 \end{array} \right]^{\mathsf{T}}$ is feasible since $\overline{x}_{\mathcal{B}} \geqslant 0$. It corresponds to the vertex $(0, 0)$ of a polytope given by the original problem.

The set $\mathcal{B} = \{2, 4\}$ is basic. The corresponding basic solution $\overline{x}_{\mathcal{B}} = \left[\begin{array}{cccc} 0 & 8 & 0 & -21 \end{array} \right]^{\mathsf{T}}$ is infeasible since $\overline{x}_{\mathcal{B}} \not\geqslant 0$. The basic set $\mathcal{B} = \{2, 4\}$ is infeasible.

Basic Feasible Solution

Let $\mathcal{B} = \{x_{i_1}, \ldots, x_{i_m}\}$ be a basic set. Let $x_{\mathcal{B}} = \left[\begin{array}{cccc} x_{i_1} & x_{i_2} & \cdots & x_{i_m} \end{array}\right]^{\mathsf{T}}$ and let $x_{\mathcal{D}} = \left[\begin{array}{cccc} x_{j_1} & x_{j_2} & \cdots & x_{j_{n-m}} \end{array}\right]^{\mathsf{T}}$, where $\{j_1, j_2, \ldots, j_{n-m}\} = \{1, 2, \ldots, n\} - \mathcal{B}$ and $j_1 < j_2 < \ldots < j_{n-m}$. Moreover, let $A_{\mathcal{B}}$ be a submatrix of A consisting of columns $i_1, \ldots, i_m$ and let $A_{\mathcal{D}}$ be submatrix of A consisting of columns $j_1, \ldots, j_{n-m}$. Then

$$Ax = b \iff A_{\mathcal{B}}x_{\mathcal{B}} + A_{\mathcal{D}}x_{\mathcal{D}} = b \iff x_{\mathcal{B}} + A_{\mathcal{B}}^{-1}A_{\mathcal{D}}x_{\mathcal{D}} = A_{\mathcal{B}}^{-1}b.$$

Therefore, the basic solution is given by $x_{\mathcal{D}} = \left[\begin{array}{ccc} 0 & \ldots & 0 \end{array}\right]^{\mathsf{T}}$ and $x_{\mathcal{B}} = A_{\mathcal{B}}^{-1}b$. This means a basic solution can be computed by performing elementary row operations on the matrix $[A|b]$ until the columns $i_1, \ldots, i_m$ will be equal to $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$, respectively.

Example

Consider a linear programming problem
$c^\mathsf{T} x \longrightarrow min,\ Ax = b,\ x \geqslant 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{2, 4\}$ is basic. We compute the basic solution by using elementary row operations on $[A|b]$ to get the $2-$nd column equal to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the $4-$th column equal to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\left[\begin{array}{rrrr|r} 2 & 1 & 1 & 0 & 8 \\ -1 & 3 & 0 & 1 & 3 \end{array}\right] \xrightarrow{r_2 - 3r_1} \left[\begin{array}{rrrr|r} 2 & 1 & 1 & 0 & 8 \\ -7 & 0 & -3 & 1 & -21 \end{array}\right]$$

Therefore if $x_1 = x_3 = 0$ (non-basic variables) then
$x_2 = 8,\ x_4 = -21$ (basic variables). Since $x_4 < 0$ the basic solution
$\overline{x}_{\mathcal{B}} = \begin{bmatrix} 0 & 8 & 0 & -21 \end{bmatrix}^\mathsf{T}$ is infeasible.

Next Lecture - Simplex Method

We will learn an algorithm, called simplex method, for finding an optimal solution. Simplex method starts from a basic feasible set and with each turn moves to another basic feasible solution decreasing the objective function.