Linear Algebra

Lecture 11 - Affine Space $\mathbb{R}^n$

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Affine Space

Definition

Let $V \subset \mathbb{R}^n$ be a subspace and let $p \in \mathbb{R}^n$. An **affine space** passing through p and parallel to V is the set

$$E = p + V = \{p + v \mid v \in V\}.$$

The associated vector space of E is called the **direction** of E and is denoted $\overrightarrow{E} = V$. Elements of E are called **points** and the dimension of E is defined to be the dimension of V, i.e. $\dim E = \dim V$.

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Note that there is no distinguished point in an affine space and any affine space E is invariant under translations by vectors from $\overrightarrow{E}$.

Example

Let $p = (1, -1)$ and $V = \text{lin}((2, 3)) \subset \mathbb{R}^2$. Then

$$E = p + V = \{(1 + 2t, -1 + 3t) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}.$$

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Example

Let $p = (1, 1, 1), q = (1, 2, 3)$. Then $\overrightarrow{pq} = q - p = (0, 1, 2)$.

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Example

Let $p = (1, 1, 1), q = (1, 2, 3)$. Then $\overrightarrow{pq} = q - p = (0, 1, 2)$.

Remark

Note that if $p, q \in E$ then $p + V = q + V$. Since $q = p + \overrightarrow{pq}$ then $q + V = p + \overrightarrow{pq} + V = p + V$.

Affine Combination

Definition

Let $p_0, \ldots, p_k \in \mathbb{R}^n$ be points. For any $a_i \in \mathbb{R}$ such that $\sum_{i=0}^{k} a_i = 1$ the point $\sum_{i=0}^{k} a_i p_i$ is called the **affine combination** of $p_0, \ldots, p_k$.

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Proposition

Let $E = p + V$ be an affine space in $\mathbb{R}^n$. Then any affine combination of $p_0, \ldots, p_k \in E$ belongs to E.

Proof.

Let $p_i = p + v_i$, where $v_i \in V$ for $i = 0, \ldots, k$. Then $\sum_{i=0}^k a_i p_i = p + \sum_{i=0}^k a_i v_i \in E$. □

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$$\mathrm{af}(p_0, \ldots, p_k) = \{\sum_{i=0}^{k} a_i p_i \in \mathbb{R}^n \mid \sum_{i=0}^{k} a_i = 1\},$$

the set of all affine combinations of $p_0, \ldots, p_k$. It is the smallest affine space in $\mathbb{R}^n$ containing $p_0, \ldots, p_k$.

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Proposition

Let $p_0, \ldots, p_k \in \mathbb{R}^n$. Then

$$\mathrm{af}(p_0, \ldots, p_k) = p_0 + \mathrm{lin}(\overrightarrow{p_0 p_1}, \ldots, \overrightarrow{p_0 p_k}).$$

Proof.

Let $\sum_{i=0}^{k} a_i = 1$. Then $\sum_{i=0}^{k} a_i p_i = p_0 + \sum_{i=0}^{k} a_i(p_i - p_0) = p_0 + \sum_{i=0}^{k} a_i \overrightarrow{p_0 p_i} \in p_0 + \operatorname{lin}(\overrightarrow{p_0 p_1}, \ldots, \overrightarrow{p_0 p_k})$.

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Assume $p = p_0 + \sum_{i=1}^{k} \alpha_i \overrightarrow{p_0 p_k}$. Then

$p = (1 - \sum_{i=1}^{k} \alpha_i) p_0 + \sum_{i=1}^{k} \alpha_i p_k$. $\square$

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Example

$$\mathrm{af}((1,1,1),(1,2,3),(3,2,1)) = (1,1,1) + \mathrm{lin}((0,1,2),(2,1,0)).$$

Proof.

Let $\sum_{i=0}^{k} a_i = 1$. Then $\sum_{i=0}^{k} a_i p_i = p_0 + \sum_{i=0}^{k} a_i(p_i - p_0) = p_0 + \sum_{i=0}^{k} a_i \overrightarrow{p_0 p_i} \in p_0 + \mathrm{lin}(\overrightarrow{p_0 p_1}, \dots, \overrightarrow{p_0 p_k})$.

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$$\overrightarrow{p_0 p_1} = (0,1,2),$$

$$\overrightarrow{p_0 p_2} = (2,1,0).$$

Parametrization

Definition

Let $E = p + \mathrm{lin}(v_1, \dots, v_k) \subset \mathbb{R}^n$ where vectors $v_1, \dots, v_k$ are linearly independent. Then any point $q \in E$ can be uniquely written as

$$q = p + \sum_{i=1}^{k} t_i v_i.$$

Any such presentation of E is called a **parametrization**.

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$$= (1,2,3) + \text{lin}((0,1,2),(1,1,1))$$

that is $(1 + 2t_2, 1 + t_1 + t_2, 1 + 2t_1)$, $t_1, t_2 \in \mathbb{R}$ and $(1 + t_2, 2 + t_1 + t_2, 3 + 2t_1 + t_2)$, $t_1, t_2 \in \mathbb{R}$ are two different parametrizations of E.

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Any affine space E in $\mathbb{R}^n$ is equal to a set of solutions of a system of linear equations in n variables.

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Proposition

Any affine space E in $\mathbb{R}^n$ is equal to a set of solutions of a system of linear equations in n variables.

Proof.

There exists a homogeneous system of linear equations describing the vector space $\overrightarrow{E}$

$$\overrightarrow{E}: \begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = 0 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = 0 \end{cases}$$

Proof.

Let $E = p + \overrightarrow{E}$. If $p = (y_1, \dots, y_n)$ set

$$\begin{array}{ccccccccc} b_1 & = & a_{11}y_1 & + & a_{12}y_2 & + & \dots & + & a_{1n}y_n \\ b_2 & = & a_{21}y_1 & + & a_{22}y_2 & + & \dots & + & a_{2n}y_n \\ \vdots & & & \vdots & & \ddots & & \vdots & \vdots \\ b_m & = & a_{m1}y_1 & + & a_{m2}y_2 & + & \dots & + & a_{mn}y_n \end{array}$$

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Then the affine space E is described by

$$E : \left\{ \begin{array}{cccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = b_m \end{array} \right.$$

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$\square$

The constants $b_1, \dots, b_m$ do not depend on the point $p \in E$ since any two points in E differ by a vector from $\overrightarrow{E}$.

Examples

Example

Describe by a system of linear equations an affine space E parallel to $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ passing through $p = (2, 3, 4)$.

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Example

Describe by a system of linear equations the affine space $E = p + V$ in $\mathbb{R}^4$ where

$$p = (1, 1, 2, 1), \;\; V = \mathrm{lin}((1, 1, 3, 0), (1, 0, 1, 0), (0, 1, 2, 0)).$$

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Vectors $(1, 0, 1, 0), (0, 1, 2, 0)$ form a basis of V. Therefore V is described by the system of equations

$$V: \begin{cases} x_1 + 2x_2 - x_3 = 0 \\ x_4 = 0 \end{cases}$$

Examples (continued)

Example

Recall $E = (1, 1, 2, 1) + V$. Therefore

$$E: \begin{cases} x_1 + 2x_2 - x_3 = 1 \\ x_4 = 1 \end{cases}$$

Definition

For any $p, q \in \mathbb{R}^n$ the **distance** between p and q is $\|\overrightarrow{pq}\|$. It is denoted $d(p, q)$.

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For any $p, q \in \mathbb{R}^n$ the **distance** between p and q is $\|\overrightarrow{pq}\|$. It is denoted $d(p,q)$.

It has the following properties:

i) $d(p,q) \geqslant 0$ and $(d(p,q) = 0 \iff p = q)$,

ii) $d(p,q) = d(q,p)$ (symmetry),

iii) $d(p,r) \leqslant d(p,q) + d(q,r)$ (triangle inequality).

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The affine space $\mathbb{R}^n$ equipped with a function satisfying above properties (called metric) becomes a **metric space**.

Affine Transformation

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Definition

Let $E \subset \mathbb{R}^n, H \subset \mathbb{R}^m$ be two affine spaces. A function $f : E \longrightarrow H$ satisfying the condition

$$f(p + \alpha) = f(p) + f'(\alpha)$$

for some $p \in E$ and some linear transformation $f' : \overrightarrow{E} \longrightarrow \overrightarrow{H}$ is called an **affine transformation**.

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If $q \in E$ then $f(q + \alpha) = f(p + \overrightarrow{pq} + \alpha) = f(p) + f'(\overrightarrow{pq}) + f'(\alpha) =$ $= f(q) + f'(\alpha)$ therefore the condition in the definition holds for any $p \in E$.

Properties

Proposition

Let E, H be two affine spaces. Then $f : E \longrightarrow H$ is an affine transformation if and only if

$$f(\sum_{i=0}^{k} a_i p_i) = \sum_{i=0}^{k} a_i f(p_i),$$

for any $p_i \in E$ and $a_i \in \mathbb{R}$ such that $\sum_{i=0}^{k} a_i = 1$.

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Proof.

Assume that f is an affine transformation. Then
$f(\sum_{i=0}^{k} a_i p_i) = f(p_0 + \sum_{i=0}^{k} a_i \overrightarrow{p_0 p_i}) = f(p_0) + \sum_{i=0}^{k} a_i f'(\overrightarrow{p_0 p_i}) =$
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Proof.

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We omit the proof of the other direction $\square$

Properties

Remark

Any affine transformation $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ can be written as

$$f((x_1, x_2, \ldots, x_n)) = (a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_1, \ldots,$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + b_m),$$

where $a_{ij}, b_k \in \mathbb{R}$. The linear transformation f' has matrix

$$M(f')^{st}_{st} = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$$

in standard bases.

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in standard bases.

Proof.

Choose $p = (0, \dots, 0)$ so
$f((x_1, \dots, x_n)) = f((0, \dots, 0)) + f'((x_1, \dots, x_n))$. □

Affine Orthogonal Projection and Reflection

Definition

Let $E \subset \mathbb{R}^n$ be an affine space and let $p_0 \in E$. The affine transformation $\pi_E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$\pi_E(p) = \pi_E(p_0 + \overrightarrow{p_0 p}) = p_0 + P_{\overrightarrow{E}}(\overrightarrow{p_0 p}),$$

where $P_{\overrightarrow{E}}$ is the (linear) orthogonal projection on $\overrightarrow{E}$, is called an **(affine) orthogonal projection** on E.

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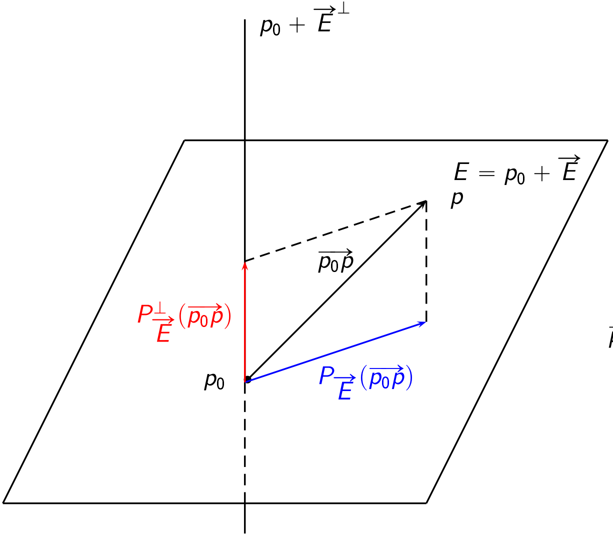
The transformation $\sigma_E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$\sigma_E(p) = \sigma_E(p_0 + \overrightarrow{p_0 p}) = p_0 + S_{\overrightarrow{E}}(\overrightarrow{p_0 p}),$$

where $S_{\overrightarrow{E}}$ is the (linear) orthogonal reflection about $\overrightarrow{E}$, is called an **(affine) orthogonal reflection** about E.

Orthogonal Projection



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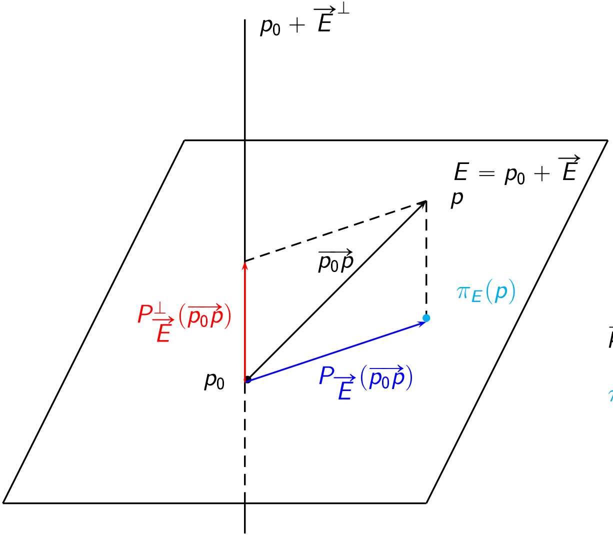

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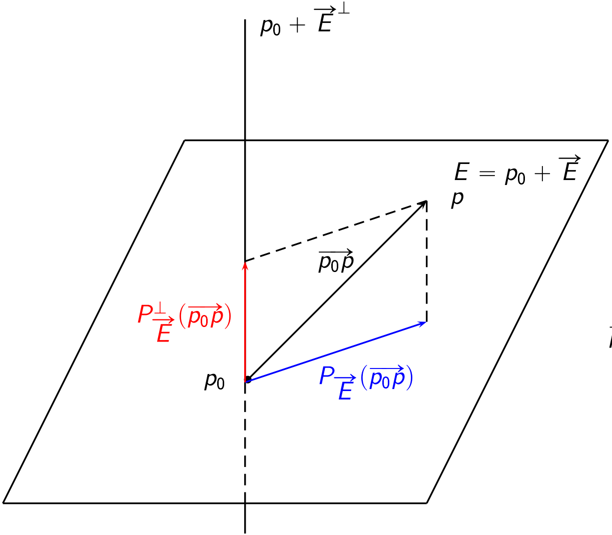
$$\pi_E(p) = p_0 + P_{\overrightarrow{E}}(\overrightarrow{p_0p})$$

Orthogonal Reflection



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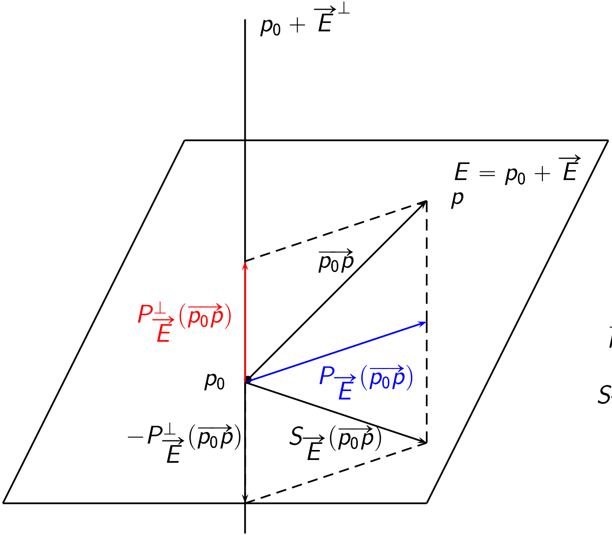

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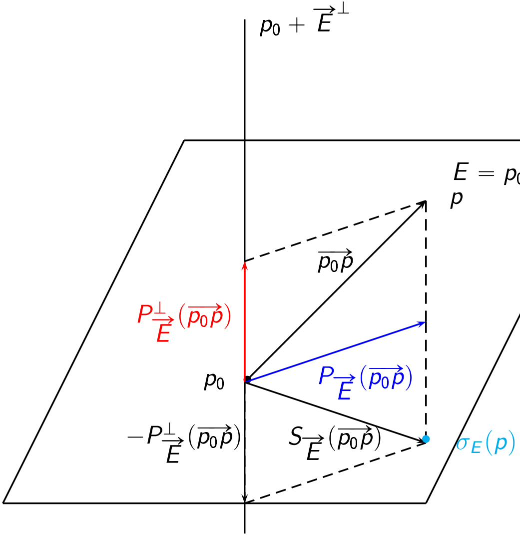

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$$\sigma_E(p) = p_0 + S_{\overrightarrow{E}}(\overrightarrow{p_0p})$$

Example

Let $p_0 = (1,1,1), p_1 = (1,2,3)$. Let $E = \mathrm{af}(p_0, p_1)$ be an affine line. Compute orthogonal projection of $p = (2,0,1)$ on E.

$$\overrightarrow{p_0 p} = (2,0,1) - (1,1,1) = (1,-1,0), \quad \overrightarrow{E} = \mathrm{lin}((0,1,2)),$$

The linear projection of $\overrightarrow{p_0 p}$ on $\overrightarrow{E}$ is

$$P_{\overrightarrow{E}}(\overrightarrow{p_0 p}) = \frac{(1,-1,0)\cdot(0,1,2)}{0^2+1^2+2^2}(0,1,2) = -\frac{1}{5}(0,1,2).$$

Therefore $\pi_E(p) = (1,1,1) - \frac{1}{5}(0,1,2) = \frac{1}{5}(5,4,3)$.

Intersection of Affine Spaces

Proposition

Let $E = p + V, H = q + W \subset \mathbb{R}^n$ *be two affine spaces. Then either* $E \cap H = \varnothing$ *or* $p_0 \in E \cap H$ *and*

$$E \cap H = p_0 + (V \cap W).$$

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Proof.

If $p_0 \in E \cap H$ then $E = p_0 + V$ and $H = p_0 + W$. $\square$

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Proposition

Let $E = p + V, H = q + W \subset \mathbb{R}^n$ *be two affine spaces. Then* $E \cap H \neq \emptyset$ *if and only if*

$$\overrightarrow{pq} = v + w, \text{ where } v \in V, w \in W.$$

Intersection of Affine Spaces (continued)

Proof.

Assume $\overrightarrow{pq} = v + w$ as above. Then $q - w \in H$ and $q - w = p + \overrightarrow{pq} - w = p + v \in E$.

Intersection of Affine Spaces (continued)

Proof.

Assume $\overrightarrow{pq} = v + w$ as above. Then $q - w \in H$ and $q - w = p + \overrightarrow{pq} - w = p + v \in E$. Assume that $p_0 \in E \cap H$. Then $\overrightarrow{pq} = \overrightarrow{pp_0} + \overrightarrow{p_0q}$ where $\overrightarrow{pp_0} \in V$ and $\overrightarrow{p_0q} \in W$. $\square$

Projection as Intersection

Proposition

Let $V \subset \mathbb{R}^n$ be a vector space. For any $p, q \in \mathbb{R}^n$ the affine spaces $p + V$ and $q + V^\perp$ intersect in exactly one point.

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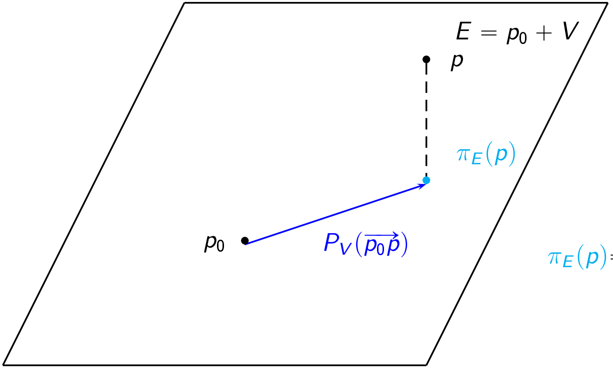
Proof.

We know $\overrightarrow{p_0p} = P_V(\overrightarrow{p_0p}) + P_{V^\perp}(\overrightarrow{p_0p})$. As in the previous proof the only point of the intersection is equal to $p_0 + P_V(\overrightarrow{p_0p})$. This is equal to $\pi_E(p)$ by definition. □

Orthogonal Projection (again)



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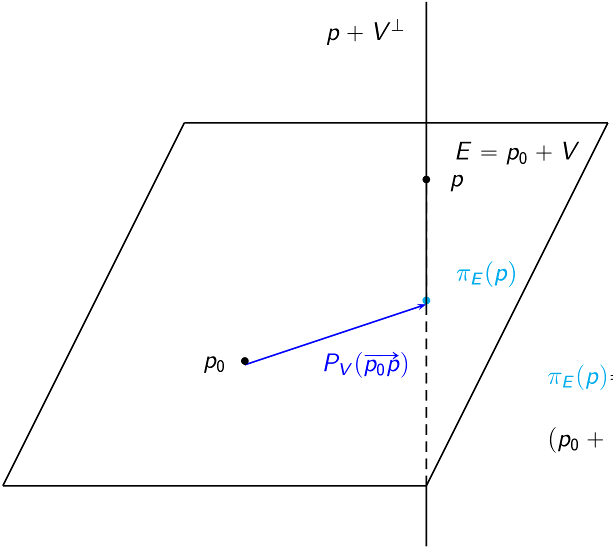

$$\pi_E(p) = p_0 + P_V(\overrightarrow{p_0 p})$$

$$(p_0 + V) \cap (p + V^{\perp}) = \pi_E(p)$$

Example

Let $p_0 = (1, 1, 1), p_1 = (1, 2, 3)$. Let $E = \mathrm{af}(p_0, p_1)$ be an affine line. Compute orthogonal projection of $p = (2, 0, 1)$ on E.

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$$E = \{(1,1,1) + t(0,1,2) \mid t \in \mathbb{R}\}.$$

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The orthogonal complement to $\overrightarrow{E}$ is two-dimensional hence given by a single equation $x_2 + 2x_3 = 0$. The point p satisfies the equation, therefore $p + \overrightarrow{E}^{\perp}$ is described by $x_2 + 2x_3 = 2$. By substituting the parametrization to the equation we get

$$(1+t) + 2(1+2t) = 2 \Longrightarrow t = -\frac{1}{5}.$$

Hence $\pi_E(2,0,1) = (1,1,1) - \frac{1}{5}(0,1,2) = \frac{1}{5}(5,4,3)$.

Example

Find a formula of a orthogonal projection onto the affine space $E = \mathrm{af}((1,1,1,1),(1,0,1,0),(1,1,0,0)) \subset \mathbb{R}^4$. The space E can be written as $E = (1,1,1,1) + \mathrm{lin}((0,1,0,1),(0,0,1,1))$. We need to find an orthogonal basis of $\overrightarrow{E}$. Set $v_1 = (0,1,0,1), v_2 = (0,0,1,1)$. Then

$$w_1 = v_1 = (0,1,0,1),$$

$$w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = (0,0,1,1) - \frac{1}{2}(0,1,0,1) = \frac{1}{2}(0,-1,2,1).$$

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$$\pi_E(x_1,x_2,x_3,x_4) = (1,1,1,1) + P_{\overrightarrow{E}}(x_1-1,x_2-1,x_3-1,x_4-1) =$$

$$= (1,1,1,1) + \frac{x_2+x_4-2}{2}(0,1,0,1) + \frac{-x_2+2x_3+x_4-2}{6}(0,-1,2,1) =$$

Example (continued)

$$\pi_E(x_1, x_2, x_3, x_4) = (1,1,1,1) + P_{\overrightarrow{E}}(x_1 - 1, x_2 - 1, x_3 - 1, x_4 - 1) =$$

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$$= (1, \frac{2x_2 - x_3 + x_4 + 1}{3}, \frac{-x_2 + 2x_3 + x_4 + 1}{3},$$

$$\frac{x_2 + x_3 + 2x_4 - 1}{3}).$$