Linear Algebra

Lecture 9 - Diagonalizable Matrices and Application

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Diagonal Matrix

Definition

The matrix $A = [a_{ij}] \in M(n \times n; \mathbb{R})$ is called **diagonal** if $a_{ij} = 0$ for any $i \neq j$, i.e.

$$A = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}.$$

Example

The matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

are diagonal.

Diagonal Matrix of Linear Endomorphism

Proposition

Let $\varphi\colon V \longrightarrow V$ be an endomorphism of vector space V and let $\mathcal{A} = (v_1, \ldots, v_n)$ be an ordered basis of V. Then $M(\varphi)_{\mathcal{A}} = [a_{ij}]$ is diagonal if and only if v_i is an eigenvector of φ. Moreover, in such case eigenvector v_i is associated to the eigenvalue a_{ii}, i.e. $\varphi(v_i) = a_{ii}v_i$.

Proof.

($\Leftarrow$) Assume each v_i is an eigenvector of φ associated to eigenvalue α_i. Then

$$\varphi(v_i) = \alpha_i v_i = 0v_1 + 0v_2 + \ldots + 0v_{i-1} + \alpha_i v_i + 0v_{i+1} + \ldots + 0v_n,$$

i.e. in the i-th column of the matrix $M(\varphi)_{\mathcal{A}}$ there is α_i in the i-th row and $0's$ elsewhere.
($\Rightarrow$) similar to the above $\square$

Example

Let $\varphi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $\varphi((x_1, x_2)) = (8x_1 + 10x_2, -3x_1 - 3x_2)$. Then

$$M(\varphi)_{st} = \begin{bmatrix} 8 & 10 \\ -3 & -3 \end{bmatrix}, \quad w_\varphi(\lambda) = \det \begin{bmatrix} 8-\lambda & 10 \\ -3 & -3-\lambda \end{bmatrix},$$

The characteristic polynomial is $w_\varphi(\lambda) = (8-\lambda)(-3-\lambda) + 30 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$. There are two eigenvalues $\lambda_1 = 2, \lambda_2 = 3$. In order to get corresponding eigenspaces solve

$$V_{(2)}: \begin{bmatrix} 6 & 10 \\ -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff x_1 = -\frac{5}{3}x_2,$$

i.e. $V_{(2)} = \{(-\frac{5}{3}x_2, x_2) \in \mathbb{R}^2 \mid x_2 \in \mathbb{R}\} = \mathrm{lin}((-5, 3))$

$$V_{(3)}: \begin{bmatrix} 5 & 10 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff x_1 = -2x_2,$$

i.e. $V_{(3)} = \{(-2x_2, x_2) \in \mathbb{R}^2 \mid x_2 \in \mathbb{R}\} = \mathrm{lin}((-2, 1))$

Example (continued)

Recall, $\varphi((x_1, x_2)) = (8x_1 + 10x_2, -3x_1 - 3x_2)$.
The basis $\mathcal{A} = ((-5, 3), (-2, 1))$ of $\mathbb{R}^2$ consists of eigenvectors and

$$M(\varphi)_{\mathcal{A}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

since

$$\varphi((-5, 3)) = 2(-5, 3) + 0(-2, 1),$$

$$\varphi((-2, 1)) = 0(-5, 3) + 3(-2, 1).$$

Eigenvectors for Different Eigenvalues

Theorem

Let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ be pairwise distinct eigenvalues of the linear endomorphism $\varphi \colon V \longrightarrow V$. Let $\mathcal{A}_i \subset V_{(\alpha_i)}$ be a finite set of linearly independent eigenvectors of φ associated to α_i for $i = 1, \ldots, k$. Then $\mathcal{A} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k$ is a set of linearly independent vectors.

Proof.

For simplicity we assume that $\mathcal{A}_i = \{v_i\}$, i.e. each set $\mathcal{A}_i$ contains one vector. Assume $\gamma_1 v_1 + \gamma_2 v_2 + \ldots + \gamma_k v_k = \mathbf{0}$. By applying φ to both sides we get $\alpha_1\gamma_1 v_1 + \alpha_2\gamma_2 v_2 + \ldots + \alpha_k\gamma_k v_k = \mathbf{0}$. Repeating this procedure we get a system of linear equations:

$$U \colon \left\{ \begin{array}{rcrcccrl} \gamma_1 v_1 & + & \gamma_2 v_2 & + & \ldots & + & \gamma_k v_k & = \mathbf{0} \\ \alpha_1\gamma_1 v_1 & + & \alpha_2\gamma_2 v_2 & + & \ldots & + & \alpha_k\gamma_k v_k & = \mathbf{0} \\ \alpha_1^2\gamma_1 v_1 & + & \alpha_2^2\gamma_2 v_2 & + & \ldots & + & \alpha_k^2\gamma_k v_k & = \mathbf{0} \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ \alpha_1^{k-1}\gamma_1 v_1 & + & \alpha_2^{k-1}\gamma_2 v_2 & + & \ldots & + & \alpha_k^{k-1}\gamma_k v_k & = \mathbf{0} \end{array} \right.$$

□

Vandermonde Determinant

One can check that the **Vandermonde determinant**

$$\det \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_k \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \dots & \alpha_k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \alpha_3^{k-1} & \dots & \alpha_k^{k-1} \end{bmatrix} = \prod_{1 \leqslant i < j \leqslant k} (\alpha_j - \alpha_i)$$

is non-zero and hence the system U can be brought by elementary row operations to a reduced echelon form

$$U : \begin{cases} \gamma_1 v_1 & & & & = \mathbf{0} \\ & \gamma_2 v_2 & & & = \mathbf{0} \\ & & \ddots & & \vdots \\ & & & \gamma_k v_k & = \mathbf{0} \end{cases}$$

Basis Consisting of Eigenvector

Which implies that $\gamma_1 = \gamma_2 = \ldots = \gamma_k = 0$ since all vectors v_i are non-zero. In the general case one can argue in a similar way.

Corollary

Let V be a finite dimensional vector space. Let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ be pairwise distinct eigenvalues of the linear endomorphism $\varphi\colon V \longrightarrow V$. Then

i) *if $v_1, \ldots, v_k \in V$ and $\varphi(v_i) = \alpha_i v_i$ for $i = 1, \ldots, k$ then the vectors $v_1, \ldots, v_k$ are linearly independent,*

ii) $\dim V_{(\alpha_1)} + \dim V_{(\alpha_2)} + \ldots + \dim V_{(\alpha_k)} \leqslant \dim V$,

iii) $\dim V_{(\alpha_1)} + \dim V_{(\alpha_2)} + \ldots + \dim V_{(\alpha_k)} = \dim V \iff$ *there exist a basis of V consisting of eigenvectors of $\varphi \iff$ the matrix of φ relative to some basis of V is diagonal.*

In the part iii) of the corollary the basis of V consists of the union of bases of $V_{(\alpha_i)}$ for $i = 1, \ldots, k$.

Example

Let $\varphi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be given by $\varphi((x_1, x_2, x_3)) = (2x_1 - 2x_2 + x_3, 2x_2 + x_3, 4x_3)$. Then

$$M(\varphi)_{st} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}, \qquad w_\varphi(\lambda) = (2-\lambda)^2(4-\lambda).$$

The eigenvalues of φ are 2 and 4.

$$V_{(2)}: \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff x_2 = x_3 = 0,$$

$$V_{(2)} = \{(x_1, 0, 0) \in \mathbb{R}^3 \mid x_1 \in \mathbb{R}\} = \operatorname{lin}((1, 0, 0))$$

$$V_{(4)}: \begin{bmatrix} -2 & -2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff x_1 = 0 \text{ and } x_3 = 2x_2,$$

$$V_{(4)} = \{(0, x_2, 2x_2) \in \mathbb{R}^3 \mid x_2 \in \mathbb{R}\} = \operatorname{lin}((0, 1, 2))$$

Example (continued)

$$V_{(2)} = \{(x_1, 0, 0) \in \mathbb{R}^3 \mid x_1 \in \mathbb{R}\} = \mathrm{lin}((1, 0, 0))$$

$$V_{(4)} = \{(0, x_2, 2x_2) \in \mathbb{R}^3 \mid x_2 \in \mathbb{R}\} = \mathrm{lin}((0, 1, 2))$$

$\dim V_{(2)} + \dim V_{(4)} = 1 + 1 < 3 = \dim \mathbb{R}^3$, therefore there is no basis of $\mathbb{R}^3$ such that matrix of φ relative to it is diagonal.

Diagonalizable Matrix

Corollary

Let V be a finite dimensional vector space and let $\dim V = n$. If the endomorphism $\varphi\colon V \longrightarrow V$ has n pairwise distinct eigenvalues then there exists a basis of V consisting of eigenvectors.

Definition

Let $A \in M(n \times n; \mathbb{R})$. We say the matrix A is **diagonalizable** if it is similar to a diagonal matrix, that is there exists an invertible matrix $C \in M(n \times n; \mathbb{R})$ such that the matrix $C^{-1}AC$ is diagonal.

Proposition

Matrix $A \in M(n \times n; \mathbb{R})$ is diagonalizable $\iff$ there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of the endomorphism $\varphi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by the condition $M(\varphi)_{st} = A$.
Moreover, if $\mathcal{A}$ is such basis and $C = M(id)^{st}_{\mathcal{A}}$ then the matrix $C^{-1}AC$ is diagonal.

Example

Matrix $A = \begin{bmatrix} 8 & 10 \\ -3 & -3 \end{bmatrix}$ is diagonalizable. Endomorphism $\varphi((x_1, x_2)) = (8x_1 + 10x_2, -3x_1 - 3x_2)$ has two eigenvalues 2 and 3. We have computed $V_{(2)} = \mathrm{lin}((-5, 3))$ and $V_{(3)} = \mathrm{lin}((-2, 1))$. Set $\mathcal{A} = ((-5, 3), (-2, 1))$ and $C = M(id)^{st}_{\mathcal{A}}$.

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = M(\varphi)_{\mathcal{A}} = M(id)^{\mathcal{A}}_{st} M(\varphi)^{st}_{st} M(id)^{st}_{\mathcal{A}}$$

$$C = \begin{bmatrix} -5 & -2 \\ 3 & 1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$$

Example (continued)

Matrix $A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ is not diagonalizable. There is no basis of $\mathbb{R}^3$ consisting of eigenvalues of the endomorphism $\varphi((x_1, x_2, x_3)) = (2x_1 - 2x_2 + x_3, 2x_2 + x_3, 4x_3)$.

Application

Proposition

Let $A = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$ *be a diagonal matrix. Then*

$$A^m = \begin{bmatrix} a_{11}^m & & 0 \\ & \ddots & \\ 0 & & a_{nn}^m \end{bmatrix} \text{ for any } m \in \mathbb{Z}.$$

Remark

Note that this, in general, **does not** *hold for non–diagonal matrices, for example* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ *and* $1^2 \neq 2$.

Application (continued)

Let $A = \begin{bmatrix} 8 & 10 \\ -3 & -3 \end{bmatrix}$. Compute A^n. Recall $D = C^{-1}AC$ hence $A = CDC^{-1}$. Therefore $A^n = CD^nC^{-1}$.

$$A^n = \begin{bmatrix} -5 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix} =$$

$$= \begin{bmatrix} -5 \cdot 2^n + 2 \cdot 3^{n+1} & -5 \cdot 2^{n+1} + 10 \cdot 3^{n+1} \\ 3 \cdot 2^n - 3^{n+1} & 3 \cdot 2^{n+1} - 5 \cdot 3^{n+1} \end{bmatrix}.$$

Symmetric Matrix and Minimal Polynomial

Definition

Matrix $A \in M(n \times n; \mathbb{R})$ is called symmetric if $A^\intercal = A$.

Proposition

Let $A \in M(n \times n; \mathbb{R})$ be a symmetric matrix. Then A is diagonalizable.

Moreover there exists an **orthogonal** basis of $\mathbb{R}^n$ consisting of eigenvectors of the endomorphism $M(\varphi)_{st} = A$, i.e. vectors of it are pairwise perpendicular.

Definition

Let $A \in M(n \times n; \mathbb{R})$. The minimal polynomial μ_A of the matrix A is a non-zero monic polynomial with real coefficients of the least degree such that $\mu_A(A) = \mathbf{0}$.

The minimal polynomial of A divides the characteristic polynomial of A, i.e. $\mu_A \mid w_A$.

Example

Let $A = \begin{bmatrix} 8 & 10 \\ -3 & -3 \end{bmatrix}$. Then $w_A(\lambda) = (\lambda - 2)(\lambda - 3)$ and the only monic divisors of w_A are $w_A, \lambda - 2, \lambda - 3$ and 1. Since A is not a diagonal matrix then $\mu_A = w_A$.

Let $B = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$. Then $w_B(\lambda) = (2 - \lambda)^2(4 - \lambda)$. Then only monic divisors of w_B are $-w_B, (\lambda - 2)^2, \lambda - 2, (\lambda - 2)(\lambda - 4)$, $\lambda - 4$ and 1. It can be checked that $\mu_B = -w_B$.

Criterion for Diagonalizability

Theorem

Let $A \in M(n \times n; \mathbb{R})$. Matrix A is diagonalizable if and only if the minimal polynomial factors as follows

$$\mu_A(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)\dots(\lambda - \alpha_k),$$

where $\alpha_i \in \mathbb{R}$ and $\alpha_i \neq \alpha_j$, i.e. α_i are pairwise distinct.

$$A = \begin{bmatrix} 8 & 10 \\ -3 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\mu_A(\lambda) = (\lambda - 2)(\lambda - 3),$$

$$\mu_B(\lambda) = (\lambda - 2)^2(\lambda - 4).$$

Matrix A is diagonalizable and matrix B is not diagonalizable.

Minimal Polynomials of Similar Matrices

Proposition

Let $A, B \in M(n \times n; \mathbb{R})$ *be similar matrices. Then* $\mu_A = \mu_B$.

Remark

Note that non-similar matrices can have the same minimal polynomials. For example

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

have the same minimal polynomial

$$\mu_A(\lambda) = \mu_B(\lambda) = (\lambda - 1)(\lambda - 2)$$

but A *and* B *are not similar.*