Linear Algebra

Lecture 10 - Scalar Product

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Scalar Product

Definition

A scalar product of two vectors
$v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ is the real number

$$v \cdot w = \sum_{i=1}^{n} v_i w_i.$$

Example

Let $v = (1, 0, -2, 3), w = (0, 2, 2, 1) \in \mathbb{R}^4$. Then
$v \cdot w = 1 \cdot 0 + 0 \cdot 2 - 2 \cdot 2 + 3 \cdot 1 = -1.$

Properties of Scalar Product

Let $v, v', w, w' \in \mathbb{R}^n$ and let $\alpha \in \mathbb{R}$. Then

i) $v \cdot w = w \cdot v$,

ii) $(\alpha v) \cdot w = \alpha(v \cdot w)$,

iii) $(v + v') \cdot w = v \cdot w + v' \cdot w, \; v \cdot (w + w') = v \cdot w + v \cdot w'$,

iv) $v \cdot v > 0$ for $v \neq \mathbf{0}$.

Length of a Vector

Definition

The length of a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ is the number

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}.$$

Obviously $\|v\| \geqslant 0$ and

$$\|v\| = 0 \iff v = \mathbf{0}.$$

Note that if $\alpha \in \mathbb{R}$ then $\|\alpha v\| = |\alpha| \|v\|$. In particular, if $v \neq \mathbf{0}$ then $\left\| \frac{v}{\|v\|} \right\| = 1$. The vector $\frac{v}{\|v\|}$ is called the **normalized vector** of v.

Definition

Two vectors $v, w \in \mathbb{R}^n$ are said to be **orthogonal** (or perpendicular) if $v \cdot w = 0$. We write $v \perp w$.

Pythagorean Theorem

Example

Let $v = (3,0,4), w = (0,1,0), u = (1,1,1)$. Then $\|v\| = \sqrt{3^2 + 0^2 + 4^2} = \sqrt{9+16} = 5$. The normalized vector of v is $\frac{1}{5}(3,0,4)$. Since $v \cdot w = 3 \cdot 0 + 0 \cdot 1 + 4 \cdot 0 = 0$ then $v \perp w$ but w is not orthogonal to u because $w \cdot u = 1$.

Theorem (Pythagoras)

If $v \perp w$ then $\|v+w\|^2 = \|v\|^2 + \|w\|^2$.

Proof.

$\|v+w\|^2 = (v+w)\cdot(v+w) = v \cdot v + v \cdot w + w \cdot v + w \cdot w = \|v\|^2 + \|w\|^2$. $\square$

Orthogonal Complement

Let $A \subset \mathbb{R}^n$ be any set. Let

$$A^\perp = \{w \in \mathbb{R}^n \mid w \cdot v = 0 \text{ for all } v \in A\}.$$

The set $A^\perp$ is a subspace of $\mathbb{R}^n$.

Definition

Let $V \subset \mathbb{R}^n$ be a subspace. The **orthogonal complement** of V in $\mathbb{R}^n$ is $V^\perp$.

Example

Let $V = \text{lin}((1,2)) \subset \mathbb{R}^2$. Then $V^\perp = \text{lin}((2,-1))$.

Properties

Proposition

Let $v_1, \ldots, v_k \in \mathbb{R}^n$*. Then*

$$\mathrm{lin}(v_1, \ldots, v_k)^\perp = \{v_1, \ldots, v_k\}^\perp.$$

Proof.

Set $V = lin(v_1, \ldots, v_k)$. Assume $w \in V^\perp$. Then, in particular, $w \cdot v_i = 0$, hence $V^\perp \subset \{v_1, \ldots, v_k\}^\perp$. If $w \cdot v_i = 0$ for $i = 1, \ldots, k$ then for any $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, k$
$w \cdot (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k) = \alpha_1(w \cdot v_1) + \alpha_2(w \cdot v_2) + \ldots + \alpha_k(w \cdot v_k) = 0.$

$\square$

Properties (continued)

Proposition

Let $V \subset \mathbb{R}^n$, $\dim V = k$. *Then* $\dim V^\perp = n - k$ *and* $V \cap V^\perp = \mathbf{0}$.

Proof.

Let $v_1, \dots v_k$ be a basis of V, where $v_i = (a_{i1}, a_{i2}, \dots, a_{in})$. By the above Proposition $(x_1, \dots, x_n) \in V^\perp$ if and only if it is a solution of the system of linear equations

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = 0 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{k1}x_1 & + & a_{k2}x_2 & + & \dots & + & a_{kn}x_n & = 0 \end{cases}$$

$\square$

Properties (continued)

Proof.

The rank of the matrix $\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kn} \end{bmatrix}$ is equal to k, hence by the Kronecker-Capelli theorem the dimension of the set of solutions is $n - k$. Moreover, if $w \in V \cap V^{\perp}$ then $w \cdot w = 0$ hence $w = \mathbf{0}$. □

Proposition

Let $V \subset \mathbb{R}^n$ be a subspace. Then $(V^{\perp})^{\perp} = V$.

Proof.

By the above $\dim(V^{\perp})^{\perp} = n - \dim V^{\perp} = n - (n - \dim V)$. Since $V \subset (V^{\perp})^{\perp}$ and both have the same dimension they are equal. □

Example

Let $V \subset \mathbb{R}^2$ be subspace given by the linear equation $2x_1 + 3x_2 = 0$. Then $V = \mathrm{lin}((-3, 2))$ and $V^\perp = \mathrm{lin}((2, 3))$.

This can be generalized to

Proposition

Let $V \subset \mathbb{R}^n$ be equal to the set of solutions of the system of linear equations

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = 0 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{k1}x_1 & + & a_{k2}x_2 & + & \dots & + & a_{kn}x_n & = 0 \end{cases}$$

Then

$$V^\perp = \mathrm{lin}((a_{11}, a_{12}, \dots, a_{1n}), \dots, (a_{k1}, a_{k2}, \dots, a_{kn})).$$

Proof.

Let $v_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ for $i = 1, \ldots, k$. Then

$$V = \{v_1, v_2, \ldots, v_k\}^{\perp}.$$

Hence

$$V^{\perp} = (\{v_1, v_2, \ldots, v_k\}^{\perp})^{\perp} = (\mathrm{lin}(v_1, v_2, \ldots, v_k)^{\perp})^{\perp} =$$

$$= \mathrm{lin}(v_1, v_2, \ldots, v_k).$$

$\square$

Example

Let $V \subset \mathbb{R}^4$ be equal to the set of solutions of the system

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 + 6x_4 = 0 \\ x_1 - 2x_2 + 5x_3 = 0 \end{cases}$$

Then $V^{\perp} = \mathrm{lin}((2, 3, 4, 6), (1, -2, 5, 0))$.

Orthogonal Basis

Definition

Let $\mathcal{A} = (v_1, \dots, v_n)$ be a basis of $\mathbb{R}^n$. The basis $\mathcal{A}$ is said to be **orthogonal** if $v_i \perp v_j$ for $i \neq j$ and $i, j = 1, \dots, n$. The basis $\mathcal{A}$ is said to be **orthonormal** if it is orthogonal and $\|v_i\| = 1$ for $i = 1, \dots, n$, i.e. each vector is of length 1.

Examples

i) the standard basis $\varepsilon_1 = (1, 0, 0, \dots, 0), \varepsilon_2 = (0, 1, 0, \dots, 0), \dots, \varepsilon_n = (0, 0, 0, \dots, 1)$ of $\mathbb{R}^n$ is orthonormal,

ii) the basis $(-1, 2, 2), (2, -1, 2), (2, 2, -1)$ is an orthogonal basis of $\mathbb{R}^3$ (but not orthonormal),

iii) the basis $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}), (\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ is an orthonormal basis of $\mathbb{R}^3$.

Coordinates Relative to Orthonormal Basis

Proposition

Let $v_1, \ldots, v_k$ be an orthogonal basis of the subspace $V \subset \mathbb{R}^n$. For any $v \in V$

$$v = \frac{v \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{v \cdot v_2}{v_2 \cdot v_2} v_2 + \ldots + \frac{v \cdot v_k}{v_k \cdot v_k} v_k.$$

Proof.

There exist unique $\alpha_i \in \mathbb{R}$ such that $v = \alpha_1 v_1 + \ldots + \alpha_k v_k$.
Therefore

$$v \cdot v_i = \alpha_1 (v_1 \cdot v_i) + \ldots + \alpha_i (v_i \cdot v_i) + \ldots + \alpha_k (v_k \cdot v_i) = \alpha_i (v_i \cdot v_i),$$

since $v_i \cdot v_j = 0$ for $i \neq j$. $\square$

Existence of Orthogonal Basis

Example

The coordinates of the vector $(1, 1, 1)$ relative to the orthogonal basis $(-1, 2, 2), (2, -1, 2), (2, 2, -1)$ of $\mathbb{R}^3$ are $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ since $\frac{(1,1,1)\cdot(-1,2,2)}{(-1,2,2)\cdot(-1,2,2)} = \frac{1}{3}$, $\frac{(1,1,1)\cdot(2,-1,2)}{(2,-1,2)\cdot(2,-1,2)} = \frac{1}{3}$, $\frac{(1,1,1)\cdot(2,2,-1)}{(2,2,-1)\cdot(2,2,-1)} = \frac{1}{3}$, i.e.

$$(1, 1, 1) = \frac{1}{3}(-1, 2, 2) + \frac{1}{3}(2, -1, 2) + \frac{1}{3}(2, 2, -1).$$

Proposition

Any subspace $V \subset \mathbb{R}^n$ has an orthogonal basis.

Proof.

A proof will be given later. $\square$

Example

Example

Let $V \subset \mathbb{R}^3$ be given by the equation $x_1 + x_2 + x_3 = 0$. We compute inductively an orthogonal basis of V by choosing vectors orthogonal to the previously chosen ones. Let $v_1' = (1, 0, -1)$. To find $v_2' \in V$ such that $v_1' \perp v_2'$ solve

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 \qquad\;\; - x_3 = 0 \end{cases} \iff \begin{cases} 2x_1 + x_2 \qquad\;\; = 0 \\ x_1 \qquad\;\; - x_3 = 0 \end{cases}$$

$\iff x_2 = -2x_1,\ x_3 = x_1$. For example $v_2' = (1, -2, 1)$. Since $\dim V = 2$ vectors v_1', v_2' form an orthogonal basis of V. By taking normalized vectors we get an orthonormal basis $\frac{1}{\sqrt{2}}(1, 0, -1), \frac{1}{\sqrt{6}}(1, -2, 1)$ of V.

Orthogonal Decomposition

Proposition

Let $V \subset \mathbb{R}^n$ be a subspace. Then any vector $w \in \mathbb{R}^n$ can be written uniquely as

$$w = v + v^{\perp} \text{ where } v \in V, \; v^{\perp} \in V^{\perp}.$$

Proof.

Let $v_1, \ldots, v_k$ be a basis of V and let $v_{k+1}, \ldots, v_n$ be a basis of $V^{\perp}$. Then

$$\alpha_1 v_1 + \ldots + \alpha_n v_n = \mathbf{0} \iff \begin{cases} \alpha_1 v_1 + \ldots + \alpha_k v_k = \mathbf{0} \\ \alpha_{k+1} v_{k+1} + \ldots + \alpha_n v_n = \mathbf{0} \end{cases} \iff$$

$$\iff \alpha_1 = \ldots = \alpha_n = 0,$$

hence $\mathcal{B} = (v_1, \ldots, v_n)$ is a basis of $\mathbb{R}^n$. This proves the existence of a decomposition.

Orthogonal Decomposition (continued)

Proof.

If

$$w = v + v^{\perp} = u + u^{\perp},$$

where $v, u \in V,\ v^{\perp}, u^{\perp} \in V^{\perp}$, then

$$v - u = u^{\perp} - v^{\perp} \in V \cap V^{\perp} = \{\mathbf{0}\}.$$

Therefore

$$v = u,\ v^{\perp} = u^{\perp}.$$

□

Orthogonal Projection and Reflection

Definition

For any subspace $V \subset \mathbb{R}^n$ the function $P_V \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$P_V(w) = v, \text{ where } w = v + v^{\perp}, v \in V, \; v^{\perp} \in V^{\perp},$$

is a linear transformation called **the orthogonal projection** on the subspace V.

Note that with the above notation $P_{V^{\perp}}(w) = v^{\perp}$, that is $w = P_V(w) + P_{V^{\perp}}(w)$.

Orthogonal Projection and Reflection (continued)

Definition

For any subspace $V \subset \mathbb{R}^n$ the function $S_V \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$S_V(w) = v - v^\perp, \text{ where } w = v + v^\perp, v \in V, \; v^\perp \in V^\perp,$$

is a linear transformation called **the orthogonal reflection** across the subspace V.

Note that

$$S_V(w) = P_V(w) - P_{V^\perp}(w) = 2P_V(w) - w.$$

Properties

Example

Let $V = \text{lin}(v)$. Then $P_V(w) = \frac{w \cdot v}{v \cdot v} v$.

Proposition

i) $P_V(w) \in V$ *and* $(P_v(w) = w \iff w \in V)$,

ii) *let* $d(w, V) = \min\{\|w - v\| \mid v \in V\}$ *be the distance between the vector* w *and the subspace* V. *Then* $P_V(w)$ *is the unique vector in* V *such that* $d(w, V) = \|w - P_V(w)\|$,

iii) *if* $v_1, \ldots, v_k$ *is an orthogonal basis of* V *then*

$$P_V(w) = \frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2 + \ldots + \frac{w \cdot v_k}{v_k \cdot v_k} v_k.$$

Properties (continued)

Proof.

ii) recall $w = P_V(w) + P_{V^\perp}(w)$, then for any $v \in V$, by the Pythagorean theorem $\|w - v\|^2 = \|(P_V(w) - v) + P_{V^\perp}(w)\|^2 = \|P_V(w) - v\|^2 + \|P_{V^\perp}(w)\|^2 \geqslant \|P_{V^\perp}(w)\|^2$ so the minimum is attained if $v = P_V(w)$.

iii) $w - (\frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2 + \ldots + \frac{w \cdot v_k}{v_k \cdot v_k} v_k) \in V^\perp$. $\square$

Properties (continued)

Proposition

Let $V \subset \mathbb{R}^n$ be a subspace. Then

i) $P_V \circ P_V = P_V$,

ii) $S_V \circ S_V = id_{\mathbb{R}^n}$,

iii) $P_V + P_{V^\perp} = id_{\mathbb{R}^n}$,

iv) $S_V = -S_{V^\perp}$.

Example

Let $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 - x_2 + 2x_3 - 2x_4 = 0\}$ and $w = (1, 0, 1, -1)$. Compute $P_V(w)$. By definition $V^\perp = \text{lin}((1, -1, 2, -2))$. Then $P_{V^\perp}(w) = \frac{w \cdot (1,-1,2,-2)}{1^2 + (-1)^2 + 2^2 + (-2)^2}(1, -1, 2, -2) = \frac{1}{2}(1, -1, 2, -2)$.
Hence $P_V(w) = w - P_{V\perp}(w) = (\frac{1}{2}, \frac{1}{2}, 0, 0)$.

Gram-Schmidt process

Let $v_1, \ldots, v_k$ be a basis of the subspace $V \subset \mathbb{R}^n$. **The Gram-Schmidt process** is an inductive way of computing an orthogonal basis $w_1, \ldots, w_k$ of V.

By induction

i) for $i = 1$ set
$$w_1 = v_1, \quad W_1 = \mathrm{lin}(w_1),$$

ii) for $1 < i \leqslant k$ set
$$w_i = v_i - P_{W_{i-1}}(v_i),$$
$$W_i = \mathrm{lin}(w_1, \ldots, w_i).$$

Gram-Schmidt process (continued)

Proposition (Gram-Schmidt)

With notation as above for $i = 1, \ldots, k$

i) $w_1, \ldots, w_i$ *is an orthogonal basis of* W_i*,*

ii) $W_i = \text{lin}(v_1, \ldots, v_i)$*.*

Since $W_k = V$ vectors $w_1, \ldots, w_k$ form an orthogonal basis of V. The normalized vectors $\frac{w_1}{\|w_1\|}, \ldots, \frac{w_k}{\|w_k\|}$ form an orthonormal basis of V.

Example

Let $v_1 = (1,0,0,1), v_2 = (1,1,0,0), v_3 = (0,1,1,1) \in \mathbb{R}^4$. Then
$w_1 = v_1,\ W_1 = \mathrm{lin}(w_1)$,
$w_1 = (1,0,0,1)$,
$w_2 = v_2 - P_{W_1}(v_2) = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1$,
$w_2 = (1,1,0,0) - \frac{1}{2}(1,0,0,1) = \frac{1}{2}(1,2,0,-1),\ W_2 = \mathrm{lin}(w_1, w_2)$
$w_3 = v_3 - P_{W_2}(v_3) = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2$,
$w_3 = (0,1,1,1) - \frac{1}{2}(1,0,0,1) - \frac{1}{6}(1,2,0,-1) =$
$\frac{1}{3}(-2,2,3,2), W_3 = \mathrm{lin}(w_1, w_2, w_3)$.
Therefore $(1,0,0,1), (1,2,0,-1), (-2,2,3,2)$ is an orthogonal basis of $V = \mathrm{lin}(v_1, v_2, v_3)$. Moreover $\frac{1}{\sqrt{2}}(1,0,0,1), \frac{1}{\sqrt{6}}(1,2,0,-1), \frac{1}{\sqrt{21}}(-2,2,3,2)$ is an orthonormal basis of V.

Remark

Note that $\frac{w \cdot v}{v \cdot v} v = \frac{w \cdot (\alpha v)}{(\alpha v) \cdot (\alpha v)} (\alpha v)$.