Linear Algebra

Lecture 2 - Vector Spaces

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9 October 2017

Vector Spaces

A **vector space** V (or **linear space**) over the real numbers is a set V of objects, called vectors, equipped with two operations:

i) addition of two vectors, i.e. to each pair of vectors $v, w \in V$ we associate the sum $v + w \in V$,

ii) multiplication of vectors by real numbers (scalars), i.e. to each vector $v \in V$ and a real number $\alpha \in \mathbb{R}$ we associate the product αv,

satisfying the following rules:

Vector Spaces (continued)

i) $v + w = w + v$ for any $v, w \in V$ (addition is commutative),

ii) $u + (v + w) = (u + v) + w$ for any $u, v, w \in V$ (addition is associative),

iii) there exists $\mathbf{0} \in V$ (the zero vector) such that $v + \mathbf{0} = v$ for any $v \in V$,

iv) for any $v \in V$ there exists a vector $-v \in V$ such that $v + (-v) = \mathbf{0}$,

v) $(\alpha + \beta)v = \alpha v + \beta v$ for any $\alpha, \beta \in \mathbb{R}$ and $v \in V$ (multiplication is distributive with respect to scalar addition),

vi) $\alpha(v + w) = \alpha v + \alpha w$ for any $\alpha \in \mathbb{R}$ and $v, w \in V$ (multiplication is distributive with respect to vector addition),

vii) $\alpha(\beta v) = (\alpha\beta)v$ for any $\alpha, \beta \in \mathbb{R}$ and $v \in V$ (scalar multiplication is compatible with multiplication of real numbers),

viii) $1v = v$ for any $v \in V$.

Few Facts

The following facts are direct consequences of these rules:

i) The element $\mathbf{0} \in V$ is unique. Suppose there is another $\mathbf{0}' \in V$, then $\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}'$.

ii) The element $-v \in V$ is unique. Suppose there are $v', v'' \in V$ such that $v + v' = v + v'' = \mathbf{0}$. Then $(v + v') + v' = (v + v'') + v'$ but this implies $v' = v''$.

iii) $0v = \mathbf{0}$. Consider $0v = (0 + 0)v = 0v + 0v$. Hence $\mathbf{0} = (0v + 0v) + (-0v)$, that is $\mathbf{0} = 0v$.

iv) $(-1)v = -v$. Consider $\mathbf{0} = (1 - 1)v = v + (-1)v$. But $-v$ is unique, hence $(-1)v = -v$.

You may try to prove in a similar fashion that $\alpha\mathbf{0} = \mathbf{0}$ or that $\alpha v = \mathbf{0}$ implies $\alpha = 0$ or $v = \mathbf{0}$.

Examples

i) the zero vector space $\{\mathbf{0}\}$,

ii) the n-tuple space $\mathbb{R}^n$, with addition $(x_1,\ldots,x_n)+(y_1,\ldots,y_n)=(x_1+y_1,\ldots,x_n+y_n)$, multiplication $\alpha(x_1,\ldots,x_n)=(\alpha x_1,\ldots,\alpha x_n)$ and the zero vector $\mathbf{0}=(0,\ldots,0)$, in particular $\mathbb{R}=$line, $\mathbb{R}^2=$plane, $\mathbb{R}^3=$three-dimensional space,

iii) the space $\mathbb{R}^\infty$ of infinite sequences of real numbers, with addition $(x_i)+(y_i)=(x_i+y_i)$, multiplication $\alpha(x_i)=(\alpha x_i)$ and the zero vector $\mathbf{0}=(0,0,\ldots)$,

iv) the space of real functions on any set X $\mathcal{F}(X,\mathbb{R})=\{f:X\longrightarrow\mathbb{R}\}$ with addition and multiplication defined pointwise: $(f+g)(x)=f(x)+g(x)$ and $(\alpha f)(x)=\alpha f(x)$. The zero vector is the constant function admitting 0 everywhere on X.

Subspaces

Let V be a vector space. A **subspace** W of V is a non-empty subset $W \subset V$ satisfying two conditions:

i) $v + w \in W$ for any $v, w \in W$ (subspace is closed under addition),

ii) $\alpha v \in W$ for any $\alpha \in \mathbb{R}$ and $v \in W$ (subspace is closed under scalar multiplication).

A subspace W of V is called **proper** if $W \neq V$. Any subspace is a vector space.

Examples

The set of solutions of any homogeneous system of linear equations in n unknowns is a subspace of $\mathbb{R}^n$

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = 0 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = 0 \end{cases}$$

It can be shown that any subspace of $\mathbb{R}^n$ is of that form. Every subspace contains $\mathbf{0}$. Note that the set of solutions of a non-homogeneous system of linear equations is not a subspace since it does not contain $\mathbf{0}$.

Examples (continued)

$\mathbb{R}_c^\infty = \{\text{sequences } (x_i) \text{ such that } x_i = 0 \text{ for all but finitely many } i\}$ is a subspace of $\mathbb{R}^\infty$.

Let $x_0 \in X$. Then $\{f \in \mathbb{F}(X, \mathbb{R}) \mid f(x_0) = 0\}$ is a subspace of $\mathbb{F}(X, \mathbb{R})$.

All proper subspaces of $\mathbb{R}^2$ are lines through the origin $(0, 0)$ and the zero subspace $\{(0, 0)\}$. Similarly, all proper subspaces of $\mathbb{R}^3$ are planes and lines through the origin $(0, 0, 0)$ and the zero subspace $\{(0, 0, 0)\}$.

If $U, V \subset W$ are subspaces of vector space W, then $U \cap V$ is a subspace of W. You may try to prove that $U \cup V$ is a subspace of W if and only if $U \subset V$ or $V \subset U$.

Linear Combinations

Let V be a vector space. The **linear combination** of vectors $v_1, \ldots, v_k \in V$ with coefficients $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ is the vector $\alpha_1 v_1 + \ldots + \alpha_k v_k = \sum_{i=1}^{k} \alpha_i v_i \in V$. The set of all linear combinations of vectors $v_1, \ldots, v_k$ will be denoted by $\mathrm{lin}(v_1, \ldots, v_k)$.

$$\mathrm{lin}(v_1, \ldots, v_k) = \{\alpha_1 v_1 + \ldots + \alpha_k v_k \mid \alpha_1, \ldots, \alpha_k \in \mathbb{R}\}.$$

For example, the linear combination of vectors $v_1 = (1, 0, 1), v_2 = (0, 1, 0), v_3 = (1, -1, 0)$ in $\mathbb{R}^3$ with coefficients $3, 2, 1$ is the vector $(4, 1, 3) = 3(1, 0, 1) + 2(0, 1, 0) + (1, -1, 0)$.

Linear Span

Let V be a vector space.

Proposition

If vectors $v, w \in V$ are linear combinations of vectors $v_1, \ldots, v_k \in V$ then so is $v + w$.

Proof.

Let $v = \alpha_1 v_1 + \ldots \alpha_k v_k$ and $w = \beta_1 v_1 + \ldots \beta_k v_k$. Then $v + w = (\alpha_1 + \beta_1) v_1 + \ldots (\alpha_k + \beta_k) v_k$. $\square$

Corollary

The set $\mathrm{lin}(v_1, \ldots, v_k)$ *is a subspace of V.* $\square$

If $W = \mathrm{lin}(v_1, \ldots, v_k)$ then we call W the linear **span** of the vectors $v_1, \ldots, v_k$. We say W is **spanned** by the vectors $v_1, \ldots, v_k$.

Corollary

If $w_1, \ldots, w_l \in \mathrm{lin}(v_1, \ldots, v_k)$ then $\mathrm{lin}(w_1, \ldots, w_l) \subset \mathrm{lin}(v_1, \ldots, v_k)$. $\square$

Linear Span (continued)

Let V be a vector space.

Proposition

For any $v_1, \ldots, v_k \in V$ and $\alpha \in \mathbb{R} - \{0\}$ the following hold:

i) $\mathrm{lin}(v_1, v_2, \ldots, v_k) = \mathrm{lin}(\alpha v_1, v_2, \ldots, v_k)$

ii) $\mathrm{lin}(v_1, v_2, \ldots, v_k) = \mathrm{lin}(v_1 + v_2, v_2, \ldots, v_k)$

Corollary

We have

$$\mathrm{lin}(v_1, \ldots, v_k) = \mathrm{lin}(v_1 + \alpha v_2, v_2, \ldots, v_k),$$

that is, elementary operations on vectors does not change the spanned subspace.