Linear Algebra

Lecture 5 - Operations on Linear Transformations

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Sum and Scalar Multiplication

Proposition

Let V, W be vector spaces. Let $\varphi, \psi : V \longrightarrow W$ be linear transformations and let $\alpha \in \mathbb{R}$. The transformation $\varphi + \psi : V \longrightarrow W$, defined by $(\varphi + \psi)(v) = \varphi(v) + \psi(v)$ for $v \in V$, and the transformation $\alpha\varphi$ defined by $(\alpha\varphi)(v) = \alpha\varphi(v)$ are linear. The transformation $\varphi + \psi$ is called a **sum** *of φ and ψ and $\alpha\varphi$ is called a* **product** *of the transformation φ with scalar α.*

Example

Let $\varphi, \psi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ are given by
$\varphi((x_1, x_2, x_3)) = (x_1 + 2x_2 - x_3, x_1 + 2x_2 + x_3)$ and
$\psi((x_1, x_2, x_3)) = (-x_1 + x_2 + x_3, 3x_1 - 2x_2 + x_3)$. Fix $\alpha = 2$. Then
$(\varphi + \psi)((x_1, x_2, x_3)) = (3x_2, 4x_1 + 2x_3)$ and
$(2\varphi)((x_1, x_2, x_3)) = (2x_1 + 4x_2 - 2x_3, 2x_1 + 4x_2 + 2x_3)$.

Composition

Proposition

Let U, V, W be vectors spaces and let $\varphi : U \longrightarrow V$, $\psi : V \longrightarrow W$ be linear transformations. The transformation $\psi \circ \varphi \colon U \longrightarrow W$, defined by $(\psi \circ \varphi)(v) = \psi(\varphi(v))$ for $v \in U$, is linear. It is called the **composition** *of ψ with φ.*

Example

Let $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be linear transformations given by $\varphi((x_1, x_2, x_3)) = (x_1 - x_2 + 2x_3, -x_1 + 3x_2 - x_3)$ and $\psi((y_1, y_2)) = (y_1 - y_2, y_1 + 2y_2))$. Then $(\psi \circ \varphi)((x_1, x_2, x_3)) = \psi((x_1 - x_2 + 2x_3, -x_1 + 3x_2 - x_3)) = ((x_1 - x_2 + 2x_3) - (-x_1 + 3x_2 - x_3), (x_1 - x_2 + 2x_3) + 2(-x_1 + 3x_2 - x_3)) = (2x_1 - 4x_2 + 3x_3, -x_1 + 5x_2)$.

Operation on Matrices

Definition

Let $A, B \in M(m \times n; \mathbb{R}), \alpha \in \mathbb{R},\ A = [a_{ij}], B = [b_{ij}]$. The **sum** of matrices A and B is matrix $A + B = [a_{ij} + b_{ij}]$. The **product** of matrix A by scalar α is the matrix $\alpha A = [\alpha a_{ij}]$.

Example

Let $\alpha = 2$ and let $A, B = M(2 \times 3; \mathbb{R})$ be given by

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then

$$A + B = \begin{bmatrix} 0 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix},\ \alpha A = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 2 & 0 \end{bmatrix}.$$

Matrix Multiplication

Definition

Let $A \in M(m \times n; \mathbb{R})$ and let $B \in M(n \times l; \mathbb{R})$. The **matrix product** of A by B is a matrix $AB = [c_{ij}] \in M(m \times l; \mathbb{R})$ where $c_{ij} = \sum_{s=1}^{n} a_{is} b_{sj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, l$.

In particular, if $R_i = \left[\begin{array}{cccc} a_{i1} & a_{i2} & \ldots & a_{in} \end{array} \right] \in M(1 \times n; \mathbb{R})$ is the $i-$th row of matrix A and $C_j = \left[\begin{array}{c} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{array} \right] \in M(n \times 1; \mathbb{R})$ is the $j-$th column of matrix B then $R_i C_j = \left[\begin{array}{c} a_{i1}b_{1j} + \ldots + a_{in}b_{nj} \end{array} \right]$ is a 1×1 matrix which can be identified with a real number.

Matrix Multiplication (continued)

Using this identification we can write

$$AB = \begin{bmatrix} R_1 C_1 & R_1 C_2 & \dots & R_1 C_l \\ R_2 C_1 & R_2 C_2 & \dots & R_2 C_l \\ \vdots & \vdots & \ddots & \vdots \\ R_m C_1 & R_m C_2 & \dots & R_m C_l \end{bmatrix}.$$

Example

Let $A \in M(3 \times 2; \mathbb{R})$ and $B \in M(2 \times 2; \mathbb{R})$ be given by

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} R_1 C_1 & R_1 C_2 \\ R_2 C_1 & R_2 C_2 \\ R_3 C_1 & R_3 C_3 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 5 & -4 \\ 0 & -3 \end{bmatrix}.$$

In simple terms, the first column of AB is the sum of columns of A and the second one is the first column of A minus twice the second column of A.

Warning

The matrix multiplication is, in general, not commutative. For example

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

but

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Operation on Linear Transformations and Matrices

Theorem (Addition)

Let V, W be vector spaces and let $\varphi, \psi : V \longrightarrow W$ be linear transformations. Let $\mathcal{A}, \mathcal{B}$ be bases of V and W respectively. Then $M(\varphi + \psi)_{\mathcal{A}}^{\mathcal{B}} = M(\varphi)_{\mathcal{A}}^{\mathcal{B}} + M(\psi)_{\mathcal{A}}^{\mathcal{B}}$.

Theorem (Composition and multiplication)

Let U, V, W be vectors spaces and let $\varphi : U \longrightarrow V$, $\psi : V \longrightarrow W$ be linear transformations. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be the bases of U, V and W, respectively. Then $M(\psi \circ \varphi)_{\mathcal{A}}^{\mathcal{C}} = M(\psi)_{\mathcal{B}}^{\mathcal{C}} M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$.

Example (continued)

Let $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be linear transformations given by $\varphi((x_1, x_2, x_3)) = (x_1 - x_2 + 2x_3, -x_1 + 3x_2 - x_3)$ and $\psi((y_1, y_2)) = (y_1 - y_2, y_1 + 2y_2))$. Recall that $(\psi \circ \varphi)((x_1, x_2, x_3)) = (2x_1 - 4x_2 + 3x_3, -x_1 + 5x_2)$. We will compute this again, using matrix multiplication. Let $\mathcal{A}$ be the standard basis in $\mathbb{R}^3$ and let $\mathcal{B} = \mathcal{C}$ be the standard basis in $\mathbb{R}^2$. Then

$$M(\psi \circ \varphi)^{\mathcal{C}}_{\mathcal{A}} = M(\psi)^{\mathcal{C}}_{\mathcal{B}} M(\varphi)^{\mathcal{B}}_{\mathcal{A}} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & -4 & 3 \\ -1 & 5 & 0 \end{bmatrix}.$$

This agrees with the formula of $\psi \circ \varphi$.

Applications

Proposition

Let V, W *be vector spaces and let* $\varphi : V \longrightarrow W$ *be a linear transformation. Let* $\mathcal{A} = (v_1, \ldots, v_n)$ *be an ordered basis of* V *and let* $\mathcal{B} = (w_1, \ldots, w_m)$ *be an ordered basis of* W*. For any vector* $v \in V$ *let* $\alpha_1, \ldots, \alpha_n$ *be the coordinates of* v *relative to the basis* $\mathcal{A}$ *and let* $\beta_1, \ldots, \beta_m$ *be the coordinates of* $\varphi(v)$ *relative to the basis* $\mathcal{B}$*, that is* $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ *and* $\varphi(v) = \beta_1 w_1 + \ldots + \beta_m w_m$*. Then*

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}.$$

Example

Let $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformations given by $\psi((x_1, x_2)) = (x_1 - x_2, x_1 + 2x_2)$. Let $st = ((1,0),(0,1))$ be the standard basis in $\mathbb{R}^2$ and let
$\mathcal{A} = ((1,2),(0,1))$, $\mathcal{B} = ((1,0),(1,-1))$ be other two bases of $\mathbb{R}^2$. We check immediately that

$$\psi(1,2) = (-1,5) = 4(1,0) - 5(1,-1),$$

$$\psi(0,1) = (-1,2) = 1(1,0) - 2(1,-1).$$

Therefore

$$M(\psi)_{st}^{st} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \quad M(\psi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 4 & 1 \\ -5 & -2 \end{bmatrix}.$$

Pick, say, $v = (1,1)$. Since $v = 1(1,2) - 1(0,1)$, the coordinates of v relative to $\mathcal{A}$ are $1, -1$. Since $\psi(v) = (0,3) = 3(1,0) - 3(1,-1)$, the coordinates of $\psi(v)$ relative to $\mathcal{B}$ are $3, -3$.

Example (continued)

$$M(\psi)_{st}^{st} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \; M(\psi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 4 & 1 \\ -5 & -2 \end{bmatrix}.$$

the coordinates of $v = (1, 1)$ relative to the basis $\mathcal{A}$ are $1, -1$

the coordinates of $\psi(v) = (0, 3)$ relative to the basis $\mathcal{B}$ are $3, -3$

$$M(\psi)_{st}^{st} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$M(\psi)_{\mathcal{A}}^{\mathcal{B}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}.$$

Applications (continued)

Let V be a vector space. The function $\mathrm{id}_V : V \longrightarrow V$ given by $id_V(v) = v$ for any $v \in V$ is a linear transformation called **the identity**.

Corollary

Let $\mathcal{A} = (v_1, \ldots, v_n)$ and $\mathcal{B} = (w_1, \ldots, w_n)$ be two ordered bases of V. For any $v \in V$ let $\alpha_1, \ldots, \alpha_n$ be the coordinates of v relative to the basis $\mathcal{A}$ and let $\beta_1, \ldots, \beta_n$ be the coordinates of v relative to the basis $\mathcal{B}$. Then

$$M(\mathrm{id}_V)^{\mathcal{B}}_{\mathcal{A}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}.$$

The matrix $M(\mathrm{id}_V)^{\mathcal{B}}_{\mathcal{A}}$ is called a **change-of-coordinates matrix**.

Applications (continued)

Proposition

Let V, W be vector spaces and let $\varphi : V \longrightarrow W$ be a linear transformation. Let $\mathcal{A}, \mathcal{A}'$ be (ordered) bases of V and let $\mathcal{B}, \mathcal{B}'$ be (ordered) bases of W. Then

$$M(\varphi)^{\mathcal{B}'}_{\mathcal{A}'} = M(\mathrm{id}_W)^{\mathcal{B}'}_{\mathcal{B}} M(\varphi)^{\mathcal{B}}_{\mathcal{A}} M(\mathrm{id}_V)^{\mathcal{A}}_{\mathcal{A}'}.$$

Proof.

This follows directly from the fact that $\mathrm{id}_W \circ \varphi \circ \mathrm{id}_V = \varphi$ and the formula relating composition of linear transformations with matrix multiplication. $\square$

Example (continued)

Let $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation given by the formula $\psi((x_1, x_2)) = (x_1 - x_2, x_1 + 2x_2)$. Let $st = ((1,0),(0,1))$ be the standard basis of $\mathbb{R}^2$ and let $\mathcal{A} = ((1,2),(0,1))$, $\mathcal{B} = ((1,0),(1,-1))$ be other two bases of $\mathbb{R}^2$.We have already checked that

$$M(\psi)_{st}^{st} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \; M(\psi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 4 & 1 \\ -5 & -2 \end{bmatrix}$$

Let check this again using the previous Proposition. It says that

$$M(\psi)_{\mathcal{A}}^{\mathcal{B}} = M(\mathrm{id}_{\mathbb{R}^2})_{st}^{\mathcal{B}} M(\psi)_{st}^{st} M(\mathrm{id}_{\mathbb{R}^2})_{\mathcal{A}}^{st}$$

We need to compute $M(\mathrm{id}_{\mathbb{R}^2})_{st}^{\mathcal{B}}$ and $M(\mathrm{id}_{\mathbb{R}^2})_{\mathcal{A}}^{st}$.

Example (continued)

We need to compute $M(\mathrm{id}_{\mathbb{R}^2})^{\mathcal{B}}_{st}$ and $M(\mathrm{id}_{\mathbb{R}^2})^{st}_{\mathcal{A}}$. Recall that $\mathcal{A} = ((1,2),(0,1))$, $\mathcal{B} = ((1,0),(1,-1))$. Since

$$\mathrm{id}((1,2)) = 1(1,0) + 2(0,1),$$

$$\mathrm{id}(0,1) = 0(1,0) + 1(0,1),$$

we have $M(\mathrm{id}_{\mathbb{R}^2})^{st}_{\mathcal{A}} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Since

$$\mathrm{id}((1,0)) = 1(1,0) + 0(1,-1),$$

$$\mathrm{id}((0,1)) = 1(1,0) - 1(1,-1),$$

we have $M(\mathrm{id}_{\mathbb{R}^2})^{\mathcal{B}}_{st} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. Using $M(\psi)^{\mathcal{B}}_{\mathcal{A}} = M(\mathrm{id}_{\mathbb{R}^2})^{\mathcal{B}}_{st} M(\psi)^{st}_{st} M(\mathrm{id}_{\mathbb{R}^2})^{st}_{\mathcal{A}}$ one can check that

$$\begin{bmatrix} 4 & 1 \\ -5 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Elementary Matrices

Fix $\alpha \in \mathbb{R},\ n > 0$ and define the following matrices
$D_i = [d_{kl}], E_{ij} = [e_{kl}], T_{ij} = [t_{kl}], \in M(n \times n; \mathbb{R})$ as follows

i) $d_{kk} = 1$ for $k \neq i,\ d_{ii} = \alpha,\ d_{kl} = 0$ elsewhere,

ii) $e_{kk} = 1$ for $k = 1, \dots, n,\ e_{ij} = 1,\ e_{kl} = 0$ elsewhere,

iii) $t_{kk} = 1$ for $k \notin \{i, j\},\ t_{ij} = t_{ji} = 1,\ t_{kl} = 0$ elsewhere.

$$D_i = \begin{array}{cc} & \begin{array}{cccccc} & i & & & & \end{array} \\ \begin{array}{c} \\ i \\ \\ \\ \\ \\ \end{array} & \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \alpha & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \end{array}, \quad E_{ij} = \begin{array}{cc} & \begin{array}{ccccccc} & & & j & & & \end{array} \\ \begin{array}{c} \\ i \\ \\ \\ \\ \\ \\ \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \end{array},$$

$$T_{ij} = \begin{array}{cc} & \begin{array}{cccccccc} & & i & & & & j & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ \\ \\ j \\ \\ \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Elementary Matrices (continued)

Proposition

Let $A \in M(n \times m; \mathbb{R})$*. Then*

i) $D_i A =$ *matrix* A *with the* i*-th row multiplied by* α,

ii) $E_{ij} A =$ *matrix* A *with the* j*-th row added to the* i*-th row*,

iii) $T_{ij} A =$ *matrix* A *with the* i*-th and* j*-th rows switched.*

Elementary row operations correspond to multiplication by elementary matrices from the left.